FORTRAN

Non-linear Equations

A non-linear equation is any equation which includes variables with a degree other than one. Therefore, any equation involving x2, x3, x4, ... would be non-linear.

For example:

y= 3x+2 is linear, because x and y are both degree 1 (no exponent)

$y = 2x^2$ is non-linear, because x is degree 2.

Iterative Procedure

Iterative methods also known as trial and error methods, are based on the idea of successive approximations.They start with one or more initial approximations to the root and obtain a sequence of approximations by repeating a fixed sequence of steps till the solution with reasonable accuracy is obtained.Iterative methods,generally,give one root at a time.

When to terminate an Iterative procedure?

We can say that iterative procedure is continued till the required degree of accuracy in the solution is achieved.But how this degree of accuracy can be measured.

Iterative Procedure

- **Termination Criteria 1**

  Suppose that starting with x_i as the current approximation,x_{i+1} is the next approximation,the iterative procedure would terminate when the inequality is satisfied. i.e. $|f(x_{i+1})| \leq$ epsilon

  The approximation x_{i+1} will be taken as the approximate solution.

- **Termination Criteria 2**

  Terminate the iterative procedure when two successive approximations differ by an amount less than or equal to the tolerance.If x_i and x_{i+1} are two successive approximations,the iterative procedure would terminate when the inequality is satisfied i.e. the absolute error is less than or equal to the prescribed tolerance.The approximation x_{i+1} will be taken as the approximate solution.

Iterative Procedure

- **Termination Criteria 3**

  Terminate the iterative procedure when for two successive approximations,the absolute value of the ratio becomes less than or equal to the prescribed tolerance i.e. when the absolute value of the relative error becomes less than or equal to the prescribed tolerance.The approximation x_{i+1} will be taken as the approximate solution.

$$\frac{|x_{i+1} - x_i|}{|x_{i+1}|} \leq \text{epsilon} \qquad \text{for } x_{i+1} \neq 0$$

BISECTION METHOD

Bisection method is one of the simplest iterative methods.To start with,two initial approximations,say x1 and x2 such that f(x1)*f(x2)<0 which ensures that root lies between x1 and x2,are taken.The next-value,say x3,as the mid point of the interval[x1,x2] is computed.There are three possibilities that can arise:

1.) If f(x3)=0,then we have a root at x3.

2.) if f(x1) and f(x3) are of opposite sign,then the root lies in the interval(x1,x3).Thus,x2 is replaced by x3,and the new interval,which is half of the current interval,is again bisected.

3.) If f(x1) and f(x3) are of same sign,then the root lies in the interval(x3,x2).Thus,x1 is replaced by x3,and the new interval,which is half of the current interval,is again bisected.

BISECTION METHOD

Therefore,by repeating this interval bisection procedure,we keep enclosing the root in a new search interval,which is halved in each iteration.This iterative cycle terminated when the search interval becomes smaller than the prescribed tolerance or the value of the function nearly vanishes at the new x-value.

EPSILON

Epsilon is the prescribed tolerance in the required root,then the iterative cycle terminates when the absolute error becomes less than or equal to epsilon i.e.

$|x1-x2| \leq \text{epsilon}$

If the prescribed tolerance as well as interval is given,then we can find the minimum number of iterations to find the root using the formula:

$$n \geq \frac{\log(x2-x1)-\log \text{epsilon}}{\log 2}$$

This table shows that the Bisection method requires a large number of iterations to achieve a reasonable degree of accuracy for the desired root.

EPSILON

- **For example:**

Number of iterations Required					
epsilon	10^{-2}	10^{-3}	10^{-4}	10^{-5}	10^{-6}
n	7	10	14	17	20

Algorithm for Bisection Method

To find a root of f(x)=0 within a prescribed tolerance say epsilon.Given values x1 and x2 such that f(x1)Xf(x2) <0.The variable x3 is used to store the mid point of the interval.

```
Begin
        read:x1,x2               //input value for x1 and x2
        read:epsilon             //input the prescribed tolerance
        do
             set x3=(x1+x2)/2                        //compute the mid point
            if(f(x1)Xf(x3) < 0) then       //select the appropriate subinterval
                     set x2=x3
            else
                     set x1=x3
             endif
             while((| x1 – x2 | >epsilon) and (f(x3)≠0))
             write:x3, "as the approximate root"    //output the computed root
End
```

Assignment for Bisection Method

1. Given the one root of the non-linear equation $x^3-4x-9=0$ lies between 2.625 and 2.75.Find the root correct to four significant digits. Ans.2.706
2. Determine the roots correct to two decimal places using Bisection method for the below given equation:
   $x^3-x-4=0$
3. Describe the Bisection method.Suppose that one real root of an equation lies in the interval [1.5,2.2] and the permitted tolerance in the solution is 0.0001,how much iteration will be performed.
4. Find the solution of $x^4+2x^2-16x+5=0$ correct to three decimal places using bisection method.
5. One real root of the equation $e^{-x}-x=0$ lies between 0 and 1.Find the root with tolerance 0.001 using bisection method.Can you guess the number of iterations required?

Assignment for Bisection Method

6. Find $(25)^{1/2}$ using Bisection method. Ans.5
7. Find the root of the equation $x^3-x^2+x-7=0$ near x=2,using bisection method in four stages. Ans 2.1125
8. Determine the two smallest roots of the following equation correct upto three significant digits.

   $f(x)=x\sin x+\cos x=0$
9. Compute the real root of $4\sin x=e^x$,upto six iterations using bisection method. Ans 0.3672
10. Find the root of the equation $x^3-2x-5=0$ in which the difference between the two iterates is less than 0.001. Ans 2.09424

Floating point Representation of numbers

In a computer,we have two types of arithmetic operations:

1.)Integer arithmetic

It deals with numbers without fractional parts.For e.g.56,34.

2.)Floating point arithmetic

It deals with numbers with fractional parts.e.g. 0.76,0.5.

In computers each location stores only a finite number of digits.Due to this all operands in arithmetic operations have only a finite number of digits.We assume that a computer has a memory where each location can store 6 digits and has a provision to store one or more signs.In such computers one way of expressing real numbers would be to assume a fixed position for the decimal point and store all numbers with an assumed decimal point.

Floating point Representation of numbers

+ One memory location

5	4	2	7	1	3

Sign ↑ **Assume Decimal point**

In such situations we can store the least numbers as 0000.01 and the greatest as 9999.99.This range is inadequate.The motive is store maximum number of significant digits in a real number and also to increase the range of values of real numbers stored.

Normalized Floating point

The shifting of decimal point to the left of the most significant digit is known as normalization and the real numbers expressed in this type are called normalized floating point number.

In this technique,we represent a real number as a combination of a mantissa and an exponent.The mantissa lies between .1 and 1(i.e. it is less than one and greater than or equal to .1) i.e.

$.1 \leq |mantissa| \leq 1.0$

The exponent is the power of 10 which multiplies the mantissa.

for e.g.,the number $34.56 * 10^6$ is expressed as .3456E8.

Thus,in a memory location,the 6 available digits are divided into two parts.First part consists of four digits of mantissa(i.e. decimal part) and second part consists of two exponents.

Normalized Floating point

These have their own independent +ve or –ve sign.

4	3	5	6	0	2

Mantissa **Exponent**

Note:

The first digit in the mantissa should always be made non-zero by adjusting the value of exponent.If the number is .007456,then while storing we made it .7456E-2.

The adjusting of mantissa to the left until its first digit is non-zero is known as **normalization.**This process is done so that maximum number of useful digits may be preserved.

Example:

Arithmetic operations with normalized Floating Point Numbers

1.)Addition operation:

We can perform the addition operation on normalized floating point numbers if the exponents of these numbers are equal.If the exponents are unequal,then the exponent of the numbers with smaller exponent is made equal to the larger exponents and mantissa is adjusted appropriately.

Ex 1. Add .4564 E4 to .1211 E4

.4564 E4

+ .1211 E4

.5775 E4

Arithmetic operations with normalized Floating Point Numbers

Ex 2. Add .8426 E3 and .2174 E5

First,make the exponents equal.So,we will adjust number with smaller exponent i.e. .8426 E3.

The decimal point of mantissa of .8426 E3 is shifted by 2(5-3) position to the left and exponent is incremented by 2.

Thus,we get .0084 E5.Now,we can add two numbers with equal exponents.

```
  .0084 E5
 +.2174 E5
  .2258 E5
```

Arithmetic operations with normalized Floating Point Numbers

Ex 3. Add .7534 E4 and .6352 E4

.7534 E4

+.6352 E4

1.3886 E4

Since the mantissa is greater then 1 and has 5 digits,so the decimal point is shifted to the left by one position and the exponent is increased by 1.We get the normalized sum as .1388 E5.

Ex 4. Add 0.5467 E99 to 0.7254 E99

0.7254 E99

+0.5467 E99

1.2721 E99

Since the mantissa is greater than 1 and has 5 digits,so the decimal point is shifted to left by one position and the exponent is increased by 1.We get the normalized sum as .1272 E100.

Arithmetic operations with normalized Floating Point Numbers

Since the exponent part cannot store more than two digits,The number greater than the largest number which computer can handle.This is known as **Overflow condition** and the arithmetic unit will indicate an error condition.

1.)Add .4236 E3 to .7462 E7.

2.)Add .4273 E-2 and 0.5324 E-3

3.)Add .4546 E3 to .5433 E7

4.)Add .6434 E3 to .4845 E3

Arithmetic operations with normalized Floating Point Numbers

2.)Multiplication operation

We can perform the multiplication operation on normalized floating point numbers by multiplying the mantissa and adding the exponents.

Ex 1.Multiply 0.6543 E5 by 0.2255 E3

=(0.6543 * 0.2255) E (5+3)

=(0.1475<u>4465</u>) E8

Discarded

=0.1475 E8

Ex 2.Multiply 0.4523 E8 by 0.7321 E-12

=(0.3311<u>2881</u>) E-4

Discarded

Arithmetic operations with normalized Floating Point Numbers

Ex 3. 0.1112 E6 by 0.1213 E8

=(0.1112 * 0.1213)E(6+8)

=(0.0134885) E14

=0.1348 E13

Ex 4. .1112 E52 by .3323 E50

=(.1112 * .3323) E(52+50)

=(0.369517) E 102

=0.3695 E101

It is Overflow condition.

Arithmetic operations with normalized Floating Point Numbers

a.) .1111 E10 by .1234 E15

b.) .1234 E-75 by .1111 E -37

c.) 0.2345 E5 by 0.4201 E3

d.) 0.7423 E5 by 0.3122 E2

4.)Divide operation

We can perform the division operation on normalized floating point numbers by dividing the mantissa of the numerator by that of the denominator.In this division operation,denominator exponent is subtracted from the numerator exponent.

Ex 1. Divide 0.6663 E8 by 0.2000 E5

=(0.6663 / 0.2000) E(8-5)

=(3.3315) E3

=0.3331 E4

Arithmetic operations with normalized Floating Point Numbers

Ex 2. Divide 0.9998 E5 by 0.1000 E-99

Ex 3. Divide 0.9987 E-4 by 0.2000 E 99

Ex4. Divide 0.8888 E5 by 0.2000E3

Ex5. Divide 0.9997 E4 by 0.2000 E-99

Consequences of Normalized floating point representation

- The distributive laws of arithmetic do not always hold true. i.e.

  $(a+b)-c \neq (a-c)+b$.

  $a(b-c) \neq ab-ac$.

- We know that $4x=x+x+x+x$.But this equation may not hold true,when arithmetic is performed employing normalized floating point representation.

Newton Raphson Method

To determine a root of f(x)=0,f'(x) is a first order derivative of f(x),x_0 is the initial approximation,*e* is the error allowed in the result and n is the maximum number of iterations.

1.Read x_0,*e*,n

2.y_0<-f(x_0)

3.y_0'<-f'(x_0)

4.for i=1 to n in steps of 1 do

5.set x_1<-x_0-y_0/y_0'

6.set relative_error=|((x_1-x_0)/x_1)|

7.set x_0=x_1

8.if (relative_error < e)then

  write:x1,"as the approximate root"

  exit

Newton Raphson Method

Endif

Endfor

Write:”solution does not converge in”,n,”iterations”

End.

Assignment of Newton Raphson Method

1. Use Newton method of approximation to evaluate the real cube root of 7 correct to 3 places of decimal.
   Ans.1.913
2. Obtain the root of the following equation $x^3-x^2-1=0$ correct to three decimal places using Newton Raphson and Bisection Method.
   Ans.1.4655
3. Find the root of the equation 3x=cosx+1 Ans.0.6071
4. Find the root of the equation $x^2-5=0$.Perform atleast four iterations.
5. Find the root of the equation $x+\log_{10}x=3.375$ correct to three significant places. Ans.2.911
6. Compute the real roots of $x^3+12.1x^2+13.1x+22.2=0$ correct to two decimal places. Ans -11.1000

Birge-Vieta Method

This is an iterative method to find a real root of the **n**th degree polynomial equation **f(x) = Pn(x) = 0** of the form

$$a_n x^n + a_{n-1} x^{n-1} + . \ . \ . + a_1 x + a_0 = 0$$

Newton Raphson method to solve a polynomial equation f(x)=0 is called Birge-Vieta Method if synthetic division be used to find $f(x_n)$ and $f'(x_n)$ in

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

p_k	a_0	a_1	$a_2 \ldots\ldots$	a_{n-1}	a_n
		$p_k a_0$	$p_k a_1 \ldots$	$p_k a_{n-2}$	$p_k a_{n-1}$
	b_0	b_1	$b_2 \ldots\ldots$	b_{n-1}	b_n
		$p_k b_0$	$p_k b_1$	$p_k b_{n-2}$	$p_k b_{n-1}$
	c_0	c_1	c_2	c_{n-1}	

Birge-Vieta Method

Then $p_{k+1} = p_0 - \frac{b_3}{c_2}$

Assignment:

1. Find the real root of **$x^3 - x^2 - x + 1 = 0$** Ans. 1.0
2. **$x^4 - 3x^3 + 3x^2 - 3x + 2 = 0$** Ans 1.0
3. Find the root of **$x^4 - x - 10 = 0$** Ans 1.8555
4. Find the root of **$x^3 - 6x^2 + 11x - 6 = 0$** Ans 0.9999
5. Find the root of **$x^3 - 4x^2 + 5x - 2 = 0$** Ans 2.0

Birge-Vieta Method

1Find the root of $x^4 - x - 4 = 0$

2Find the root of $2x^3 - 3x^2 + 2x - 3 = 0$

3Find the root of $x^3 - 5x^2 + 4x - 3 = 0$

4Find the root of $x^3 - x^2 - x + 1 = 0$

5Find the root of $9x^4 + 30x^3 + 34x^2 + 30x + 25 = 0$

6Find the root of $x^5 - 2x^4 + 4x^3 - x^2 - 7x + 5 = 0$

Gauss Elimination

We use Gauss Elimination to solve simultaneous Linear equations.

Let Simultaneous Linear Equations are:

$a_{11}x_1+a_{12}x_2+a_{13}x_3=a_{14}$ ....R_1

$a_{21}x_1+a_{22}x_2+a_{23}x_3=a_{24}$ ....R_2

$a_{31}x_1+a_{32}x_2+a_{33}x_3=a_{34}$ …. R_3

Operate $R_2 = R_2 - \frac{a_{21}}{a_{11}} R_1$ and $R_3 = R_3 - \frac{a_{31}}{a_{11}} R_1$ and $R_4 = R_4 - \frac{a_{41}}{a_{11}} R_1$

Gauss Elimination

We get the following equations:

$$a_{11}x_1+a_{12}x_2+a_{13}x_3=a_{14} \quad \ldots\ldots R_1$$
$$a_{22}x_2+a_{23}x_3=a_{24} \quad \ldots\ldots R_2$$
$$a_{32}x_2+a_{33}x_3=a_{34} \quad \ldots R_3$$

Operate $R_3 = R_3 - \frac{a_{32}}{a_{22}} R_2$ and $R_4 = R_4 - \frac{a_{42}}{a_{22}} R_2$, we get the following equations

$$a_{11}x_1+a_{12}x_2+a_{13}x_3=a_{14} \quad \ldots\ldots R_1$$
$$a_{22}x_2+a_{23}x_3=a_{24} \quad \ldots\ldots R_2$$
$$a_{33}x_3=a_{34} \quad \ldots R_3$$

Gauss Elimination

$$R_4 = R_4 - \frac{a_{43}R_3}{a_{33}}$$

From equation R_3, $x_3 = \frac{a_{34}}{a_{33}}$

From equation R_2, $x_2 = \frac{(a_{24} - a_{23}x_3)}{a_{22}}$

From equation R_1, $x_1 = \frac{[a_{14} - (a_{13}x_3 + a_{12}x_2)]}{a_{11}}$

Gauss Elimination Assignment

1. $x_1+x_2+x_3=6$

   $3x_1+3x_2+4x_3=20$

   $2x_1+x_2+3x_3=13$

2. $x+4y-z=-5$

   $x+y-6z=-12$

   $x-y-z=4$

3. $2x+8y+2z=14$

   $x+6y-z=13$

   $2x-y+2z=5$

Gauss Elimination Assignment

4. $3x+y+2z=3$

   $2x-3y-z=-3$

   $x+2y+z=4$

5. $5x-y-2z=142$

   $x-3y-z=-30$

   $2x-y-3z=-5$

6. $10x+y+z=12$

   $x+10y+z=12$

   $x+y+10z=12$

Gauss Elimination Method

1.) $5x+y+z+w=4$

$x+7y+z+w=12$

$x+y+6z+w=-5$

$x+y+z+4w=-6$ Ans.1,2,-1,-2

2. $x+y+0.5z+w=3.5$

$-x+2y+w=-2$

$-3x+y+2z+w=-3$

$-x+2w=0$ Ans 2.161,-0.462,1.432,1.085

Gauss Elimination Method

3.) $1.2x+2.1y-1.1z+4w=6$

$-1.1x+2y+3.1z+3.9w=3.9$

$-2.1x-2.2y+3.7z+16w=12.2$

$-1.0x-2.3y+4.7z+12w=4$ Ans $-1.929, 1.260, -1.503, 1.004$

4.) $10x-7y+3z+5u=6$

$-6x+8y-z-4u=5$

$3x+y+4z+11u=2$

$5x-9y-2z+4u=7$ Ans $x=5, y=4, z=-7, u=1$

Gauss Jordan

Gauss Jordan is also known as Complete elimination method.

Let Simultaneous Linear Equations are:

$a_{11}x_1+a_{12}x_2+a_{13}x_3=a_{14}$ ....R_1

$a_{21}x_1+a_{22}x_2+a_{23}x_3=a_{24}$ ....R_2

$a_{31}x_1+a_{32}x_2+a_{33}x_3=a_{34}$ …. R_3

Operate $R_1=R_1/a_{11}$

$R_{2=} R_{2-}a_{21} R_1$

$R_{3=} R_{3-}a_{31} R_1$

we will get the equations:

Gauss Jordan

$a_{11}x_1+a_{12}x_2+a_{13}x_3=a_{14}$ ....R_1

$a_{22}x_2+a_{23}x_3=a_{24}$ ....R_2

$a_{32}x_2+a_{33}x_3=a_{34}$ …. R_3

Operate $R_2=R_2/a_{22}$

$R_{1=}\ R_{1-}a_{12}\ R_2$

$R_{3=}\ R_{3-}a_{32}\ R_2$

we will get the equations:

$a_{11}x_1+0x_2+a_{13}x_3=a_{14}$ ....R_1

$x_2+a_{23}x_3=a_{24}$ ....R_2

$a_{33}x_3=a_{34}$ …. R_3

Gauss Jordan

Operate $R_3 = R_3 / a_{33}$

$R_{1=}\ R_{1-} a_{13}\ R_3$

$R_{2=}\ R_{2-} a_{23}\ R_3$

we will get the equations:

$a_{11}x_1 + 0x_2 + 0x_3 = a_{14}$ ....R_1

$0x_1 + a_{22}x_2 + 0x_3 = a_{24}$ ....R_2

$0x_1 + 0x_2 + x_3 = a_{34}$ …. R_3

Gauss Jordan Assignment

1. $x_1+x_2+x_3=6$

   $3x_1+3x_2+4x_3=20$

   $2x_1+x_2+3x_3=13$

2. $x+4y-z=-5$

   $x+y-6z=-12$

   $x-y-z=4$

3. $2x+8y+2z=14$

   $x+6y-z=13$

   $2x-y+2z=5$

Gauss Jordan Assignment

4. $2x-2y+5z=13$

   $2x+3y+4z=20$

   $3x-y+3z=10$

5. $x+y+z=9$

   $2x-3y+4z=13$

   $3x+4y+5z=40$

6. $x+y+z=1$

   $4x+3y-z=6$

   $3x+5y+3z=4$

Gauss Seidal Method

Let Simultaneous Linear Equations are:

$a_{11}x_1+a_{12}x_2+a_{13}x_3=a_{14}$ I

$a_{21}x_1+a_{22}x_2+a_{23}x_3=a_{24}$ II

$a_{31}x_1+a_{32}x_2+a_{33}x_3=a_{34}$ …. III

Obtain the relation to find unknowns x1,x2,x3 from equations I,II,III

$x_1=\{ x_{14}-(a_{12}x_2+a_{13}x_3)\}/a_{11}$ ……….A

$x_2=\{x_{24}-(a_{21}x_1+a_{23}x_3)\}/a_{22}$ ………B

$x_3=\{x_{34}-(a_{31}x_1+a_{32}x_2)\}/a_{33}$,,,,,,,,,,,,C

Initially put $x_1=x_2=x_3=0$ in equation A.We will get x_1,x_2,x_3.Again put these in equations A,B,C.we will get new x_1,x_2,x_3.

Gauss Seidal Assignment

1. $x+y+z=9$

   $2x-3y+4z=13$

   $3x+4y+5z=40$

2. $x+y+z=1$

   $4x+3y-z=6$

   $3x+5y+3z=4$

3. $x_1+x_2+x_3=6$

   $3x_1+3x_2+4x_3=20$

   $2x_1+x_2+3x_3=13$