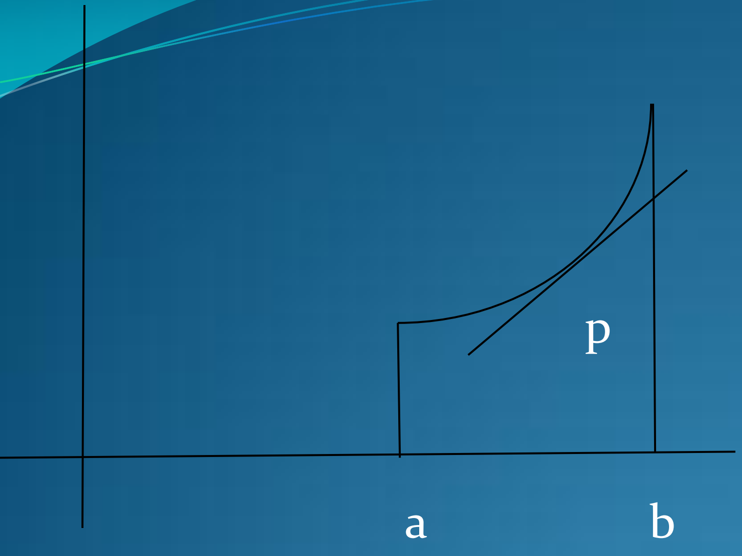
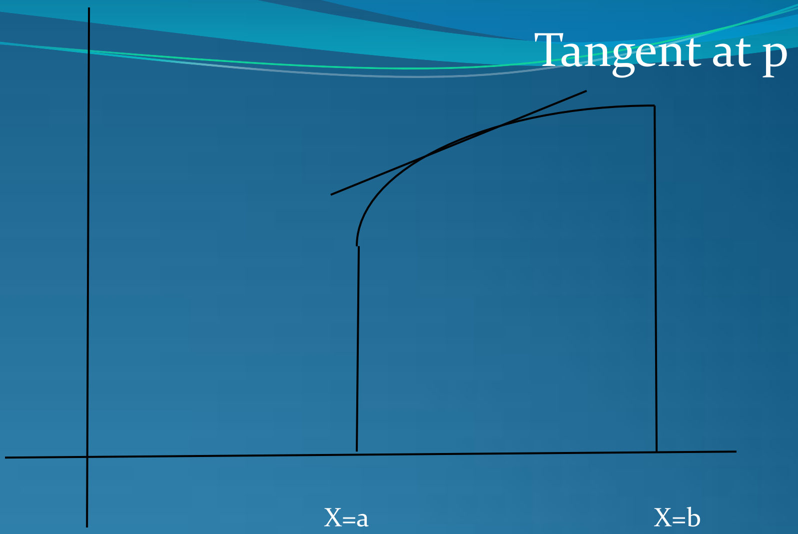
Concavity, convexity and points of inflexion

Submitted By
Vikram Kumar (maths)
P.G.G.C for Girls
Sec – 11, Chandigarh.

- Concavity
- Convexity
- Points of inflexion
- Asymptotes





Concave upward or convex downwards. A curve is said to be concave upwards(or convex downwards) on (a,b) if all the point of the curve lies above any tangent to it on that interval (see in fig.)

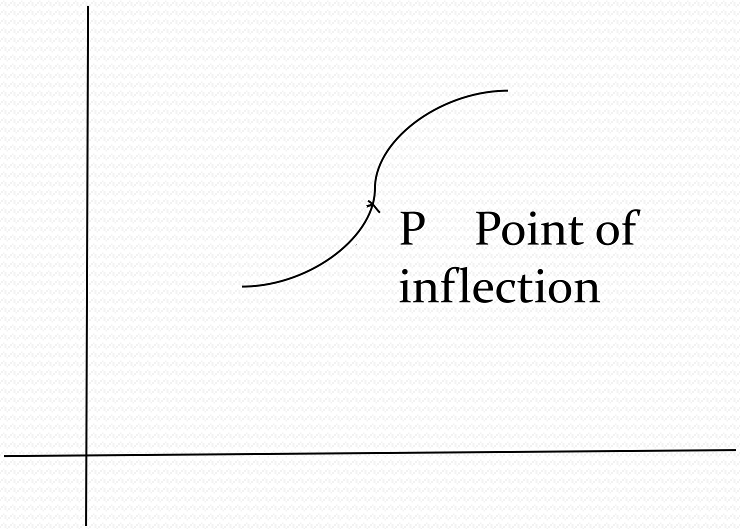
Concave downward or convex upward

A curve is said to be concave downwards (or convex upwards) on (a,b) if all the

point of the curve lies below any tangent to it on that interval (see in fig .)

Point of inflexion

A point on the curve is said to be a point of inflexion if the curve changes from concavity to convexity or from convexity to concavity in passing through the point .



Criteria for Concavity , Convexity and Inflexion

Theorem. If P(c, $f(x)$)is a point the curve y= $f(x)$ such that $f'(c)$ is finite, then

1) The curve is concave upwards(convex downwards) at P if $f''(c)>0$
2) The curve is concave downwards(convex upwards) at P if $f''(c)<0$.
3) The curve has a point of inflexion at P if $f'''(c) \neq 0$.

Another Criterion for Inflexion

Theorem. A point P(c, *f*(x)) on the curve y=*f*(x) is a point of Inflexion if $f''(x)$ changes sign as x passes through c i.e.

$\Rightarrow$ either $f''(x) < 0$, $x < c$ *and* $f''(x) > 0$; $x > c$

Or

$f''(x) > 0$, $x < c$ *and* $f''(x) < 0$; $x > c$

Important Remarks

1. The graph of f is concave up at (c , f(c)). If $f''(c) > 0$.
2. The graph of f is concave down at (c , f(c)). If $f''(c) < 0$.
3. The graph of f has inflexion at (c , f(c)).

   If $f''(c) = 0$ and $f'''(c) \neq 0$.

   Or

   If $f''(x)$ changes sign as x passes through c.
4. The graph of f may have inflexion at a point where second derivative fails to exist.

5. The point where second derivative is zero or fails to exist are possible points of inflexion. The possible point are then verified for points of inflexion by the fact that if *f"(x)* changes sign as x crosses possible point then the possible point is a point of inflexion.

Prove that

1. The graph of $f(x) = e^x$ is concave up $\forall$ $x \in R$.
2. The graph of $f(x) = log$ x is concave down or convex up $\forall$ x D_f

1. $f(x) = e^x$

$\Rightarrow f'(x) = e^x \quad \Rightarrow f''(x) = e^x > 0 \ \forall\ x \in R.$

$\Rightarrow$ graph of f *is* concave up $\forall$ $x \in R$

2. $f(x) = log(x)$; $D_f = (0, \infty)$

$\Rightarrow f'(x) = \frac{1}{x} \quad \Rightarrow f''(x) = - \left(\frac{1}{x^2} \right) < 0 \ \forall\ x \in D_f$

$\Rightarrow$ Graph of f is concave down $\forall$ $x \in D_f$

If $f(x) = ax^3 + 3bx^3$. Find a and b so that (-1 ,2) is a point of Inflection on the graph of f .

Solution :

Given $f(x) = ax^3 + 3bx^3$ (1)

$\because$ (-1 , 2) is point of inflexion on the graph f (given)

$\Rightarrow$ (-1 , 2) lies on (1) $\Rightarrow$ 2 = -a + 3b (2)

From (1) , $f'(x) = 3ax^2 + 6bx \Rightarrow f''(x) = 6ax + 6b$ exists $\forall \; x \in R$

So , possible point of inflexion are given by

$$f''(x) = 0 \Rightarrow 6ax + 6b = 0 \Rightarrow x = \frac{-b}{a} \qquad 3$$

But given (-1 , 2) is point of inflexion i.e. $x = -1$ is point of inflexion (4)

From (3) and (4) , $\frac{-b}{a} = -1 \Rightarrow b = a$ (5)

Using (5) in (2) , we have $2 = -a + 3b \Rightarrow a = 1$

Putting $a = 1$ in (5) , we have $b = 1$. Hence $a = 1$, $b = 1$

Some problem :

1. Find the point of inflexion on the curve x = $(log y)^3$

2. Discuss f(x) = (sin x + cos x) e^x , for $0 \le x \le 2\pi$ for concavity , convexity and point of inflexion.

Singular point

A point on the curve at which the curve exhibits on Extra - ordinary behavior is called a singular point.

Type of singular point

1. Point of inflexion.
2. Multiple points.

Multiple point

A point on the curve through which more than one branch of the curve pass is called a called Multiple point.

Double point

A point on the curve through which more than two branch of the curve pass is called a called Multiple point.

Note. Through a double point , two tangent in general can be drawn which may be real and different or real and coincident or imaginary.

Triple point

A point on the curve through which three branches of the curve pass is called a Triple point .

Asymptotes

A line ‘l’ at a finite distance from origin is said to be an asymptote of an infinite branch of a curve iff the line ‘l’ lies on one side of branch of curve and perpendicular distance to infinity along the branch of the curve .

Rectangular Asymptote

If an asymptote of any curve is either parallel to x – axis or parallel to y – axis then it is called rectangular asymptote.

An Asymptote parallel to x – axis is called horizontal asymptote and asymptote parallel to y – axis is called vertical asymptote.

Oblique Asymptote

If an asymptote of any curve is neither parallel to x – axis nor parallel to y – axis then it is called oblique asymptote.

Intersection of a curve and its asymptotes

Main point to be noted.

(1) The number of asymptotes of an algebraic curve of nth degree cannot exceed n .
(2) Any asymptote of a curve of nth degree cuts the curve in at the most (n - 2) points.
(3) The n asymptote of a curve of nth degree cuts the curve in at the most n(n - 2) points.

(4) If the equation of the curve is of the from $F_n + F_{n-2} = 0$ and the curve has no parallel asymptotes , then the point of intersection of the curve and its asymptotes lie on $F_{n-2} = 0$

Problem :

Find all asymptotes of the curve

$$x^4 - 2xy^3 + 2xy^3 - y^4 + x^3 - 4x^2y + 5xy^2 - 2y^3 + x^2 - 3xy + 2y^2 + 1 = 0$$