Limits and Continuity

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Limits and Continuity

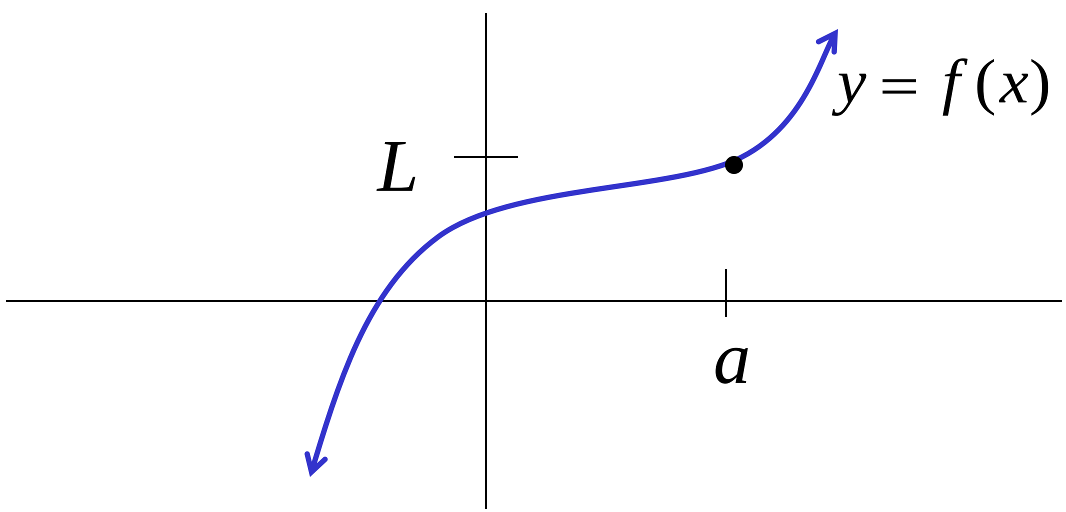
- Definition
- Evaluation of Limits
- Continuity
- Limits Involving Infinity

Limit

We say that the limit of $f(x)$ as x approaches a is L and write

$$\lim_{x \to a} f(x) = L$$

if the values of $f(x)$ approach L as x approaches a.

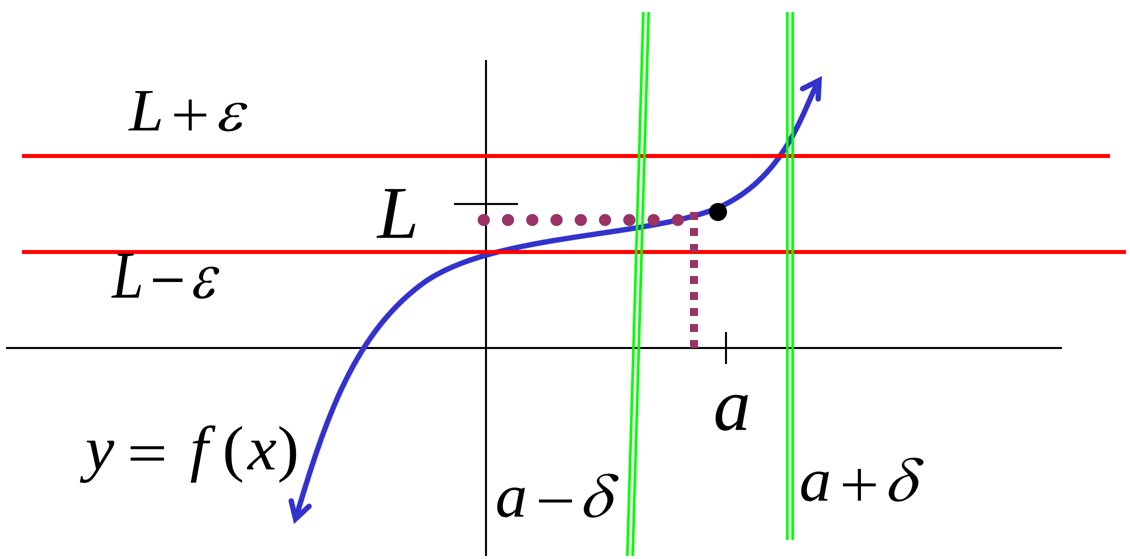


The ε-δ Definition of Limit

We say $\lim_{x \to a} f(x) = L$ if and only if

given a positive number ε, there exists a positive δ such that

if $0 < |x - a| < \delta$, then $|f(x) - L| < \varepsilon$.



Theorem

Prove that if $\underset{x \to c}{\text{Lt}} f(x)$ exists, then it is unique

Proof. If possible suppose that = l and $\underset{x \to c}{\text{Lt}} f(x) = l'$

$| l - l' | = \in > 0$

Now $\because \underset{x \to c}{\text{Lt}} f(x) = l \Rightarrow$ given $\in > 0 \exists$ a +ve number δ 1 s.t.

$| f(x) - l | < \in/2$ for $0 < | x - c | < \delta 1$

Let $\underset{x \to c}{\text{Lt}} f(x) = l'$ where $l \neq l'$

$\therefore$ for given $\in > 0$, however small $\exists$ a +ve number δ 1 δ 2

Let = Min. $\{\delta 1 \ \delta 2\}$

$\therefore$ From (1) and (2) $| f(x) - 1 | < \in/2$ for $0 < | x - c | < \delta$

and $| f(x) - 1' | < \in/2$ for $0 < | x - c | < \delta$

$| l – l' | = | (l - f(x) + (f(x) - 1') | \leq | l - f(x) | + | f(x) - 1' |$

$= | l - f(x) - 1 | + | f(x) - 1' |$

$= | f(x) - 1 | + | (f(x) - 1' |$

$< \in/2 + \in/2$

$\therefore | l – l' | < \in = | l – l' |$

Which ***is not possible***

$\therefore$ Our

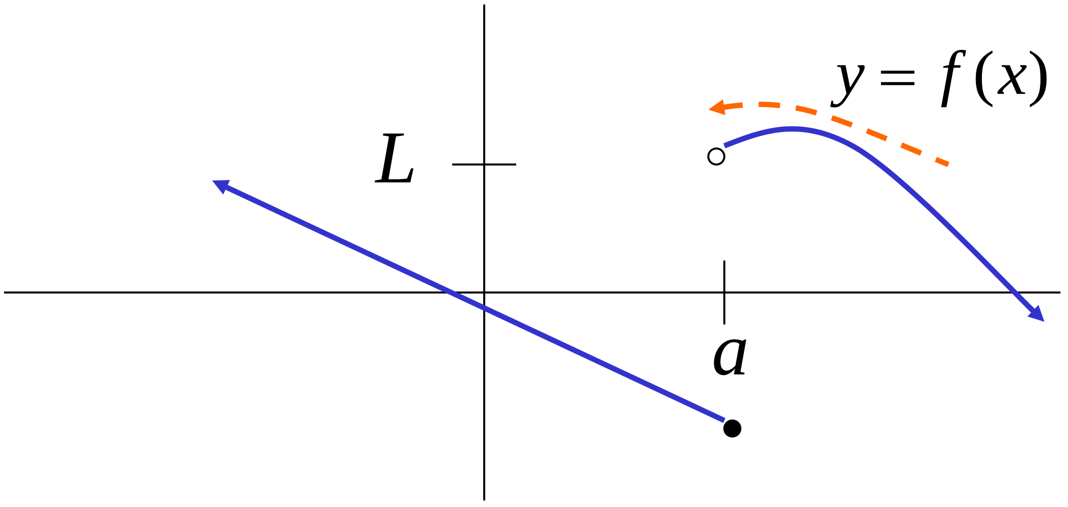
One-Sided Limit

One-Sided Limits

Function *f* is said to have left limit l ∈ R as x → c, written as Lt *f* (x) =l iff given ∈>0 however small

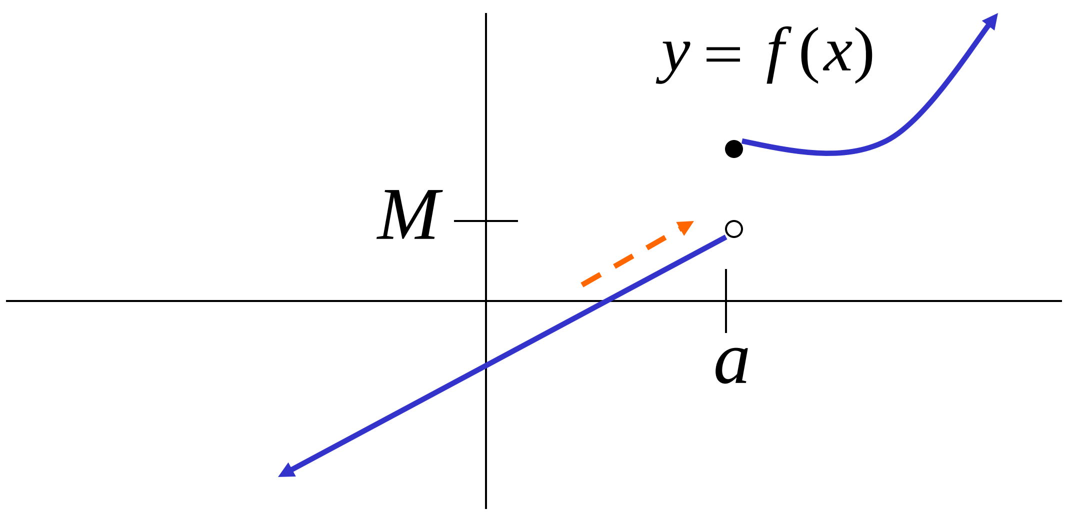
x → c⁻

there exit positive real δ depending upon ∈ such that | *f* (x) –l | < ∈ for all x in (c – δ,*c*)



Function f is said to have Right limit l ∈ R as x $\longrightarrow$ c, written as $\underset{x \to c+}{\text{Lt}} f(x) = l$ iff given ∈>0 however small

there exit positive real δ depending upon ∈ such that $|f(x) - l| < \in$ for all x in $(c, c+\delta)$



Examples

Examples of One-Sided Limit

1. Given $$f(x) = \begin{cases} x^2 & \text{if } x \le 3 \\ 2x & \text{if } x > 3 \end{cases}$$

Find $\lim_{x \to 3^+} f(x)$

$$\lim_{x \to 3^+} f(x) = \lim_{x \to 3^+} 2x = 6$$

Find $\lim_{x \to 3^-} f(x)$

$$\lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} x^2 = 9$$

More Examples

2. Let $f(x) = \begin{cases} x+1, & \text{if } x > 0 \\ x-1, & \text{if } x \le 0. \end{cases}$ Find the limits:

a) $\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} (x+1) = 0+1 = 1$

b) $\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} (x-1) = 0-1 = -1$

c) $\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (x+1) = 1+1 = 2$

d) $\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (x+1) = 1+1 = 2$

A Theorem

$\lim_{x\to a} f(x) = L$ if and only if $\lim_{x\to a^+} f(x) = L$ and $\lim_{x\to a^-} f(x) = L.$

This theorem is used to show a limit does not exist.

For the function $f(x) = \begin{cases} x+1, & \text{if } x > 0 \\ x-1, & \text{if } x \le 0. \end{cases}$

$\lim_{x\to 0} f(x)$ does not exist because $\lim_{x\to 0^+} f(x) = 1$ and $\lim_{x\to 0^-} f(x) = -1.$

But

$\lim_{x\to 1} f(x) = 2$ because $\lim_{x\to 1^+} f(x) = 2$ and $\lim_{x\to 1^-} f(x) = 2.$

Limit Theorems

If c is any number, $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M$, then

a) $\lim_{x\to a}\left(f(x)+g(x)\right) = L + M$

b) $\lim_{x\to a}\left(f(x)-g(x)\right) = L - M$

c) $\lim_{x\to a}\left(f(x)\cdot g(x)\right) = L \cdot M$

d) $\lim_{x\to a}\left(\frac{f(x)}{g(x)}\right) = \frac{L}{M}, \; (M \neq 0)$

e) $\lim_{x\to a}\left(c\cdot f(x)\right) = c \cdot L$

f) $\lim_{x\to a}\left(f(x)\right)^n = L^n$

g) $\lim_{x\to a} c = c$

h) $\lim_{x\to a} x = a$

i) $\lim_{x\to a} x^n = a^n$

j) $\lim_{x\to a}\sqrt{f(x)} = \sqrt{L}, \; (L > 0)$

Examples Using Limit Rule

Ex. $$\lim_{x\to 3}\left(x^2+1\right) = \lim_{x\to 3} x^2 + \lim_{x\to 3} 1$$

$$= \left(\lim_{x\to 3} x\right)^2 + \lim_{x\to 3} 1$$

$$= 3^2 + 1 = 10$$

Ex. $$\lim_{x\to 1}\frac{\sqrt{2x-1}}{3x+5} = \frac{\sqrt{\lim_{x\to 1}(2x-1)}}{\lim_{x\to 1}(3x+5)} = \frac{\sqrt{2\lim_{x\to 1} x - \lim_{x\to 1} 1}}{3\lim_{x\to 1} x + \lim_{x\to 1} 5}$$

$$= \frac{\sqrt{2-1}}{3+5} = \frac{1}{8}$$

More Examples

1. Suppose $\lim_{x\to 3} f(x) = 4$ and $\lim_{x\to 3} g(x) = -2$. Find

a) $$\lim_{x\to 3}\big(f(x)+g(x)\big) = \lim_{x\to 3} f(x) + \lim_{x\to 3} g(x)$$
$$= 4 + (-2) = 2$$

b) $$\lim_{x\to 3}\big(f(x)-g(x)\big) = \lim_{x\to 3} f(x) - \lim_{x\to 3} g(x)$$
$$= 4 - (-2) = 6$$

c) $$\lim_{x\to 3}\left(\frac{2f(x)-g(x)}{f(x)g(x)}\right) = \frac{\lim_{x\to 3} 2f(x) - \lim_{x\to 3} g(x)}{\lim_{x\to 3} f(x)\cdot \lim_{x\to 3} g(x)} = \frac{2\cdot 4-(-2)}{4\cdot(-2)} = \frac{-5}{4}$$

Indeterminate Forms

Indeterminate forms occur when substitution in the limit results in 0/0. In such cases either factor or rationalize the expressions.

Ex. $\lim_{x \to -5} \frac{x+5}{x^2-25}$ Notice $\frac{0}{0}$ form

$= \lim_{x \to -5} \frac{x+5}{(x-5)(x+5)}$

Factor and cancel common factors

$= \lim_{x \to -5} \frac{1}{(x-5)} = \frac{1}{-10}$

More Examples

a) $$\lim_{x\to 9}\left(\frac{\sqrt{x}-3}{x-9}\right) = \lim_{x\to 9}\left(\frac{(\sqrt{x}-3)(\sqrt{x}+3)}{(x-9)(\sqrt{x}+3)}\right)$$

$$= \lim_{x\to 9}\left(\frac{x-9}{(x-9)(\sqrt{x}+3)}\right) = \lim_{x\to 9}\left(\frac{1}{\sqrt{x}+3}\right) = \frac{1}{6}$$

b) $$\lim_{x\to -2}\left(\frac{4-x^2}{2x^2+x^3}\right) = \lim_{x\to -2}\left(\frac{(2-x)(2+x)}{x^2(2+x)}\right)$$

$$= \lim_{x\to -2}\left(\frac{2-x}{x^2}\right)$$

$$= \frac{2-(-2)}{(-2)^2} = \frac{4}{4} = 1$$

Squeeze Principle or Sandwich theorem

Statement . If $f(x) \leq g(x) \leq h(x)$ $\forall$ x in deleted nbd of c and

$\underset{x \to c}{\text{Lt}} f(x) = l = \text{Lt } h(x)$

Then lt $g(x) = l$.

Proof. $\underset{x \to c}{\text{Lt}} f(x) = l \Rightarrow$ given $\in > 0 \exists$ + ve real δ_1 s.t.

$|f(x) - l| < \in$ for $0 < |x - c| < \delta_1$ (Def.)

i.e. $l - \in < f(x) < l + \in$ for $0 < |x - c| < \delta_1$

Similarly ……..*(1)*

$\underset{x \to c}{Lt}\, h(x) = l \Rightarrow$ given $\in > 0 \exists$ +ve real δ_2 s.t.

$|h(x) - l| < \in$ for $0 < |x - c| < \delta_2$ ***(Def.)***

$l - \in < h(x) < l + \in$ for $0 < | x - c | < \delta_2$

Lt $\delta_1 = min. (\delta_1 \delta_2)$,

$\therefore$ (1) and (2) becomes

$$l - \in < f(x) < l + \in \text{ for } 0 < | x - c | < \delta \quad \ldots\ldots\ldots(3)$$

and $l - \in < h(x) < l + \in$ for $0 < | x - c | < \delta$ ………(4)

Also $f(x) \leq g(x) \leq h(x)\ \forall\ x$ *in deleted nbd of c.*

i.e. for $0 < | x - c | < \delta$ ………(5)

From (3), (4)and (5) ,we get

$$l - \in < f(x) \leq g(x) \leq h(x) < l + \in \text{ for } 0 < | x - c | < \delta$$

$\Rightarrow l - \in < g(x) < l + \in$ for $0 < | x - c | < \delta$

$\Rightarrow | g(x) - 1 | < \in$ for $0 < | x - c | < \delta$

$\Rightarrow \underset{x \to c}{Lt}\ g(x) = l$

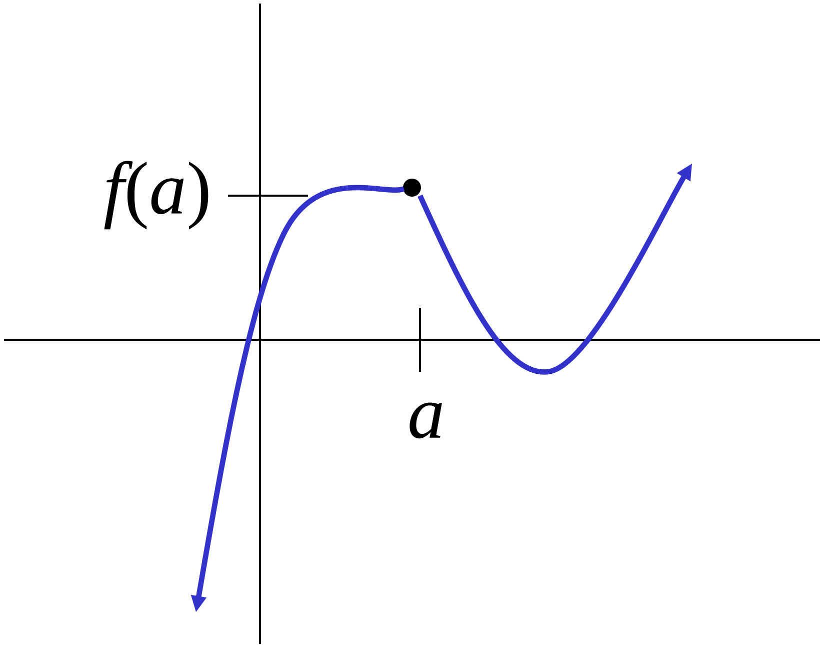
CONTINUITY

A function *f* is ***continuous*** at the point $x = a$ if the following are true:

$$i)\ \ f(a) \text{ is defined}$$

$$ii)\ \ \lim_{x \to a} f(x) \text{ exists}$$

$$iii)\ \lim_{x \to a} f(x) = f(a)$$



Examples

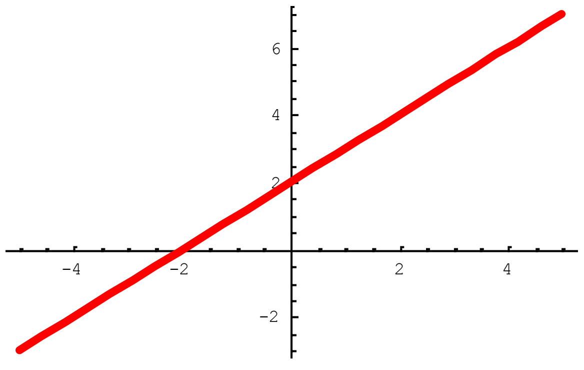
At which value(s) of x is the given function discontinuous?

1. $f(x) = x + 2$

Continuous everywhere

$$\lim_{x \to a}(x+2) = a+2$$

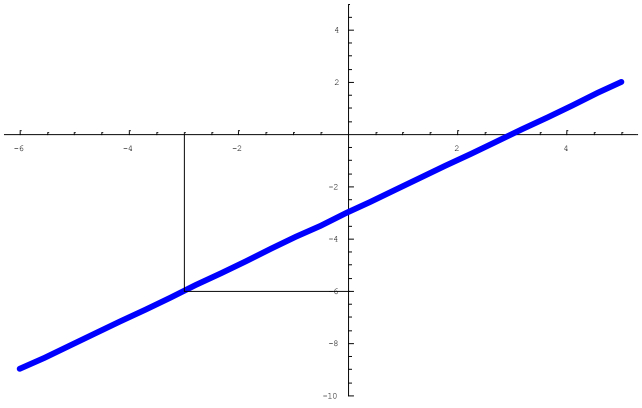
and so $\lim_{x \to a} f(x) = f(a)$



2. $g(x) = \dfrac{x^2 - 9}{x+3}$

Continuous everywhere except at $x = -3$

$g(-3)$ is undefined

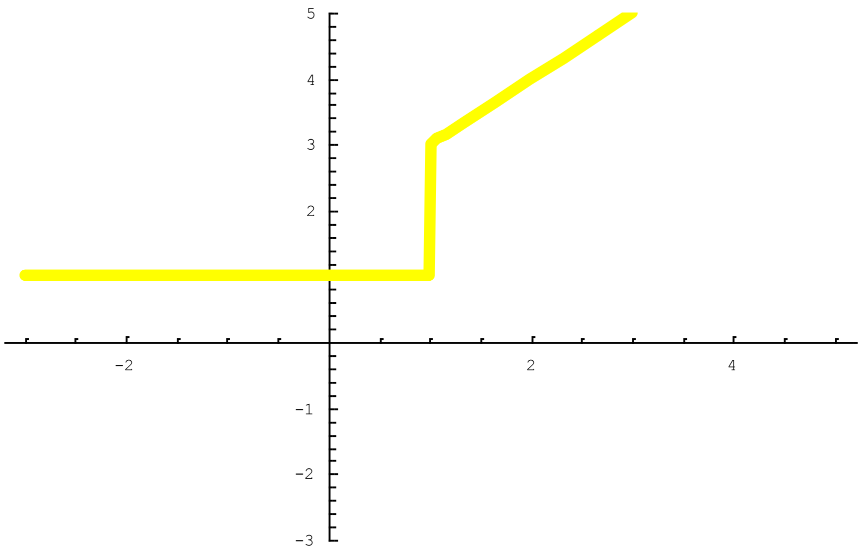


3. $h(x) = \begin{cases} x+2, \text{ if } x > 1 \\ 1, \text{ if } x \le 1 \end{cases}$

$\lim_{x \to 1^-} h(x) = 1$ and $\lim_{x \to 1^+} h(x) = 3$

Thus h is not cont. at $x=1$.

h is continuous everywhere else

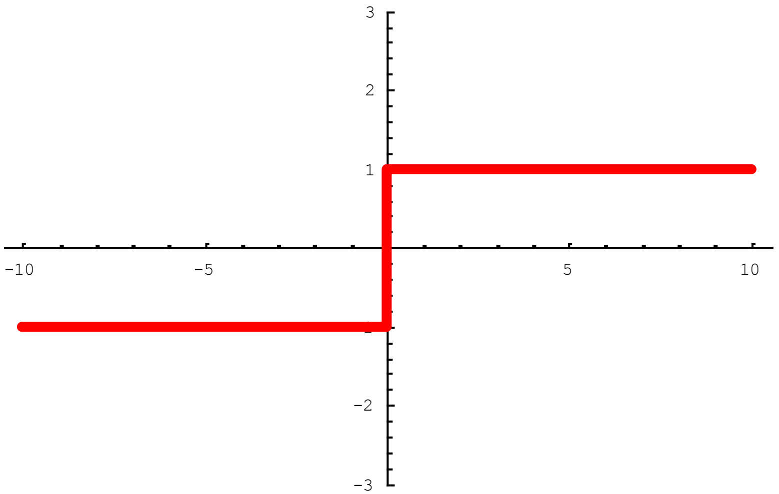


4. $F(x) = \begin{cases} -1, \text{ if } x \le 0 \\ 1, \text{ if } x > 0 \end{cases}$

$\lim_{x \to 0^+} F(x) = 1$ and $\lim_{x \to 0^-} F(x) = -1$

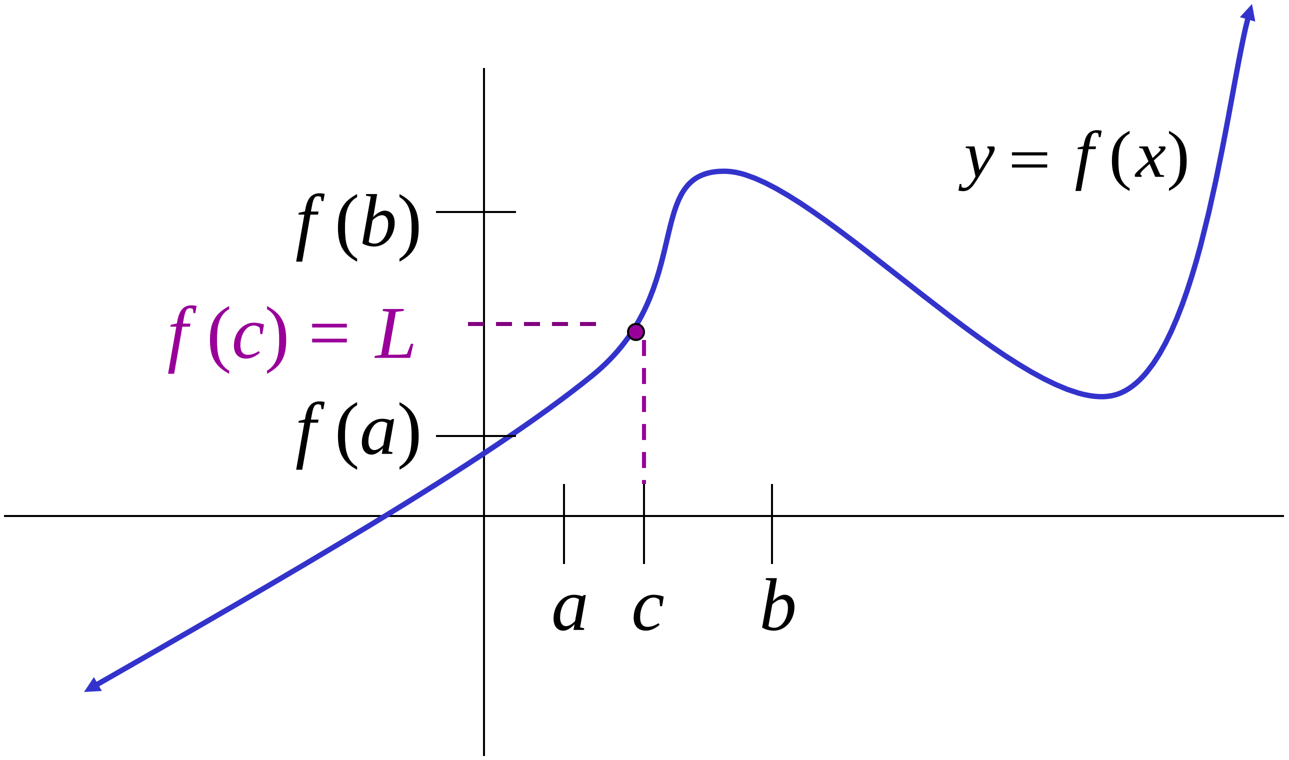
Thus F is not cont. at $x = 0$.

F is continuous everywhere else



Intermediate Value Theorem

If f is a continuous function on a closed interval $[a, b]$ and L is any number between $f(a)$ and $f(b)$, then there is at least one number c in $[a, b]$ such that $f(c) = L$.



Example

Given $f(x) = 3x^2 - 2x - 5,$

Show that $f(x) = 0$ has a solution on $[1, 2]$.

$$f(1) = -4 < 0$$

$$f(2) = 3 > 0$$

$f(x)$ is continuous (polynomial) and since $f(1) < 0$ and $f(2) > 0$, by the Intermediate Value Theorem there exists a c on [1, 2] such that $f(c) = 0$.

Limits at Infinity

For all $n > 0$, $\lim_{x\to\infty}\frac{1}{x^n} = \lim_{x\to-\infty}\frac{1}{x^n} = 0$

provided that $\frac{1}{x^n}$ is defined.

Ex. $\lim_{x\to\infty}\frac{3x^2+5x+1}{2-4x^2} = \lim_{x\to\infty}\frac{3+\frac{5}{x}+\frac{1}{x^2}}{\frac{2}{x^2}-4}$

Divide by x^2

$$=\frac{\lim_{x\to\infty}3+\lim_{x\to\infty}\left(\frac{5}{x}\right)+\lim_{x\to\infty}\left(\frac{1}{x^2}\right)}{\lim_{x\to\infty}\left(\frac{2}{x^2}\right)-\lim_{x\to\infty}4} = \frac{3+0+0}{0-4} = -\frac{3}{4}$$

More Examples

1. $$\lim_{x\to\infty}\left(\frac{2x^3-3x^2+2}{x^3-x^2-100x+1}\right)=\lim_{x\to\infty}\left(\frac{\frac{2x^3}{x^3}-\frac{3x^2}{x^3}+\frac{2}{x^3}}{\frac{x^3}{x^3}-\frac{x^2}{x^3}-\frac{100x}{x^3}+\frac{1}{x^3}}\right)$$

$$=\lim_{x\to\infty}\left(\frac{2-\frac{3}{x}+\frac{2}{x^3}}{1-\frac{1}{x}-\frac{100}{x^2}+\frac{1}{x^3}}\right)$$

$$=\frac{2}{1}=2$$

2.

$$\lim_{x\to\infty}\left(\frac{4x^2-5x+21}{7x^3+5x^2-10x+1}\right)$$

$$=\lim_{x\to\infty}\left(\frac{\frac{4x^2}{x^3}-\frac{5x}{x^3}+\frac{21}{x^3}}{\frac{7x^3}{x^3}+\frac{5x^2}{x^3}-\frac{10x}{x^3}+\frac{1}{x^3}}\right)$$

$$=\lim_{x\to\infty}\left(\frac{\frac{4}{x}-\frac{5}{x^2}+\frac{21}{x^3}}{7+\frac{5}{x}-\frac{10}{x^2}+\frac{1}{x^3}}\right)$$

$$=\frac{0}{7}$$

$$=0$$

3.

$$\lim_{x\to\infty}\left(\frac{x^2+2x-4}{12x+31}\right)$$

$$=\lim_{x\to\infty}\left(\frac{\frac{x^2}{x}+\frac{2x}{x}-\frac{4}{x}}{\frac{12x}{x}+\frac{31}{x}}\right)$$

$$=\lim_{x\to\infty}\left(\frac{x+2-\frac{4}{x}}{12+\frac{31}{x}}\right)$$

$$=\frac{\infty+2}{12}$$

$$=\infty$$

4. $$\lim_{x\to\infty}\left(\sqrt{x^2+1}-x\right)$$

$$= \lim_{x\to\infty}\left(\frac{\left(\sqrt{x^2+1}-x\right)}{1}\frac{\sqrt{x^2+1}+x}{\sqrt{x^2+1}+x}\right)$$

$$=\lim_{x\to\infty}\left(\frac{x^2+1-x^2}{\sqrt{x^2+1}+x}\right)$$

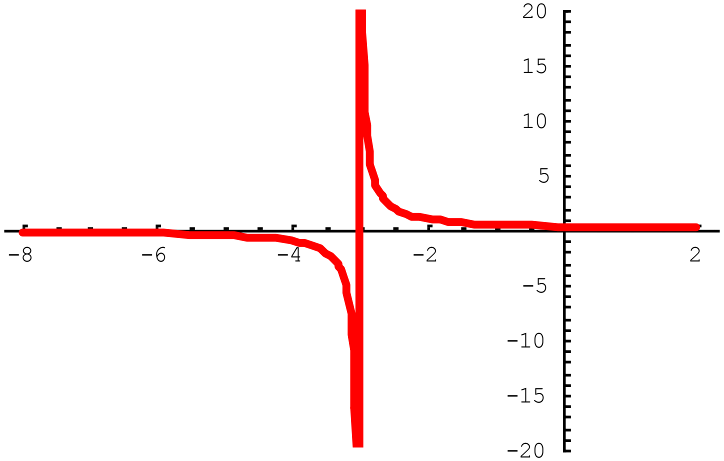
$$= \lim_{x\to\infty}\left(\frac{1}{\sqrt{x^2+1}+x}\right)$$

$$=\frac{1}{\infty+\infty}=\frac{1}{\infty}=0$$

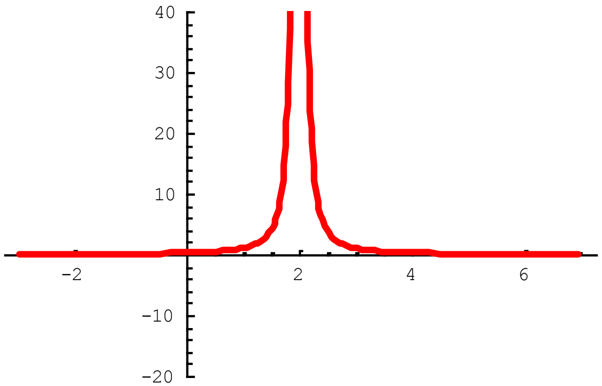
Infinite Limits

For all $n > 0$,

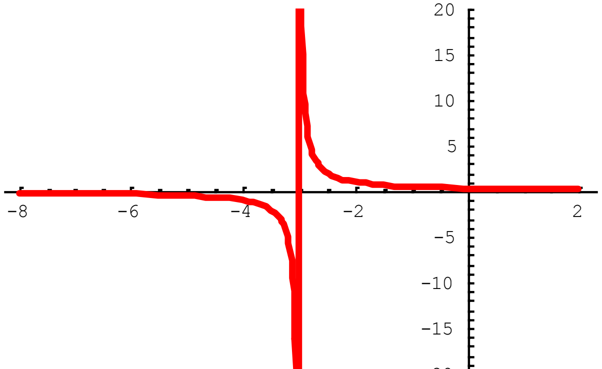
$$\lim_{x \to a^+} \frac{1}{(x-a)^n} = \infty$$



$$\lim_{x \to a^-} \frac{1}{(x-a)^n} = \infty \text{ if } n \text{ is even}$$



$$\lim_{x \to a^-} \frac{1}{(x-a)^n} = -\infty \text{ if } n \text{ is odd}$$



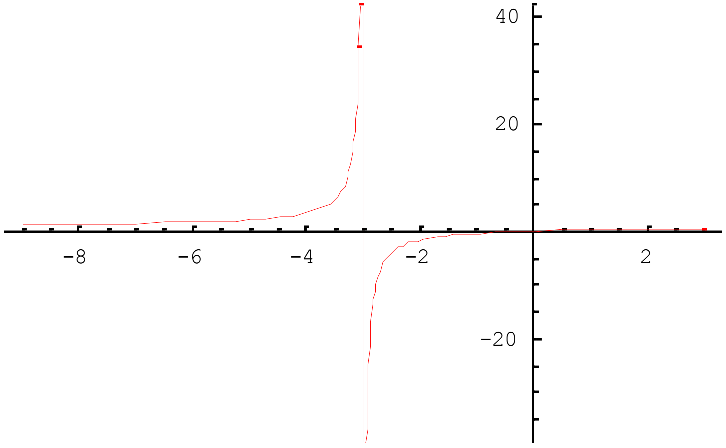
More Graphs

Examples

Find the limits

1. $$\lim_{x\to 0^+}\left(\frac{3x^2+2x+1}{2x^2}\right) = \lim_{x\to 0^+}\left(\frac{3+\frac{2}{x}+\frac{1}{x^2}}{2}\right) = \frac{3+\infty+\infty}{2} = \infty$$

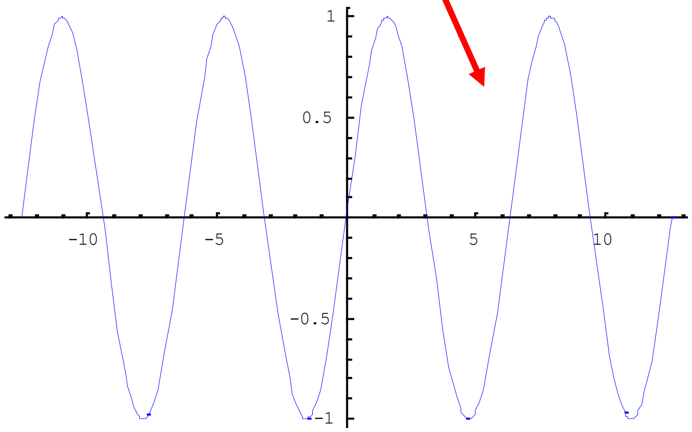
2. $$\lim_{x\to -3^+}\left(\frac{2x+1}{2x+6}\right) = \lim_{x\to -3^+}\left(\frac{2x+1}{2(x+3)}\right) = -\infty$$

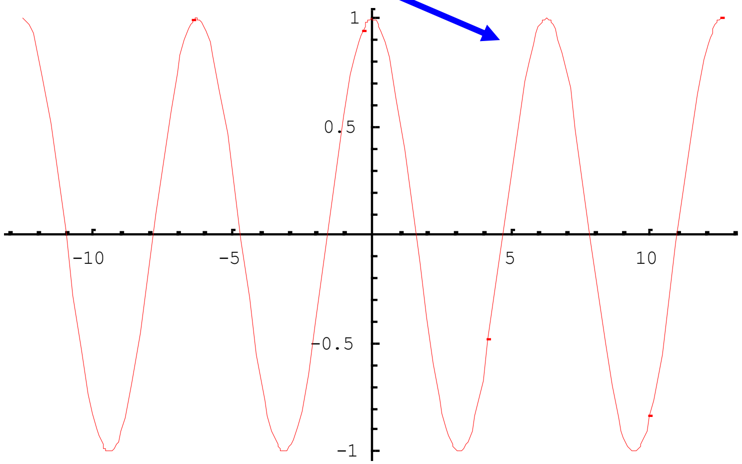


Limit and Trig Functions

From the graph of trigs functions

$$f(x) = \sin x \text{ and } g(x) = \cos x$$





we conclude that they are continuous everywhere

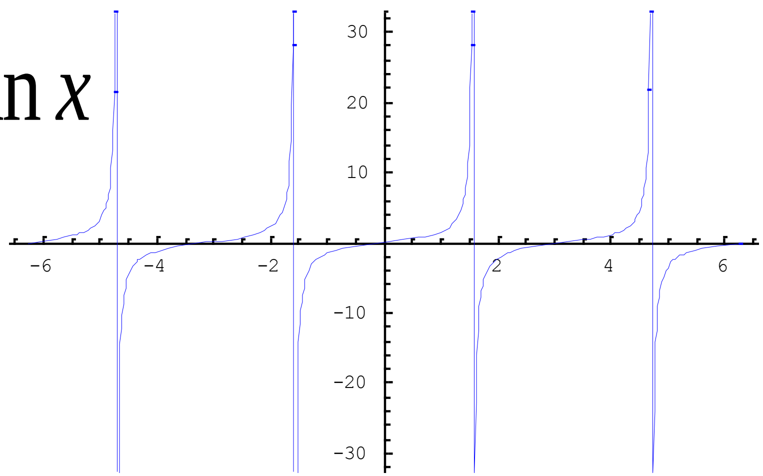
$$\lim_{x \to c} \sin x = \sin c \text{ and } \lim_{x \to c} \cos x = \cos c$$

Tangent and Secant

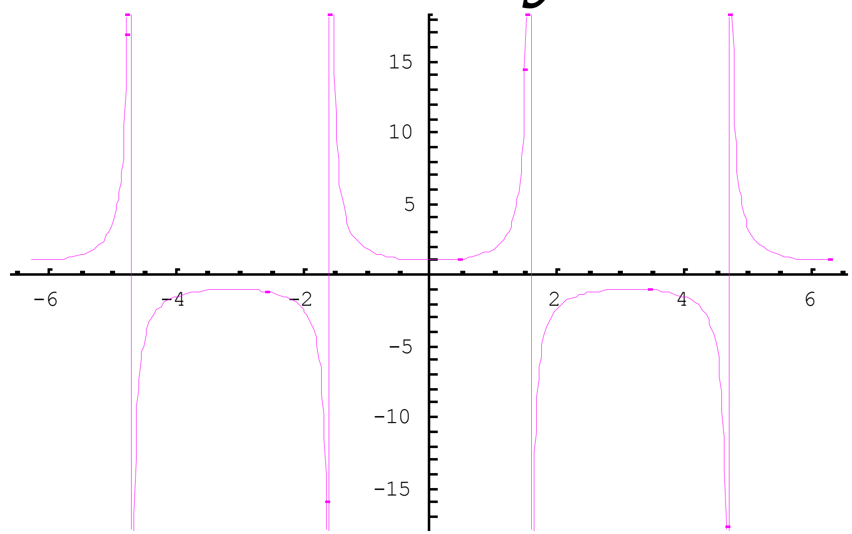
Tangent and secant are continuous everywhere in their domain, which is the set of all real numbers

$$x \neq \pm \pi/2,\ \pm 3\pi/2,\ \pm 5\pi/2,\ \pm 7\pi/2, \cdots$$

$y = \tan x$



$y = \sec x$



Examples

a) $\lim_{x\to\left(\pi/2\right)^{+}} \sec x = -\infty$

b) $\lim_{x\to\left(\pi/2\right)^{-}} \sec x = \infty$

c) $\lim_{x\to\left(-3\pi/2\right)^{+}} \tan x = -\infty$

d) $\lim_{x\to\left(-3\pi/2\right)^{-}} \tan x = \infty$

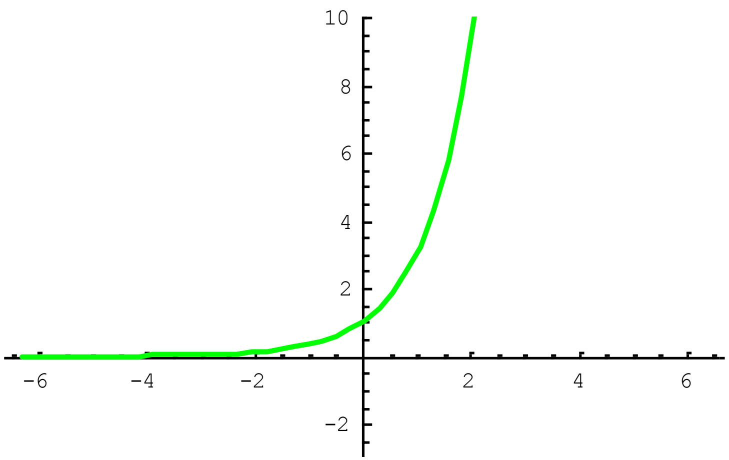
e) $\lim_{x\to\pi^{-}} \cot x = -\infty$

f) $\lim_{x\to\pi/4} \tan x = 1$

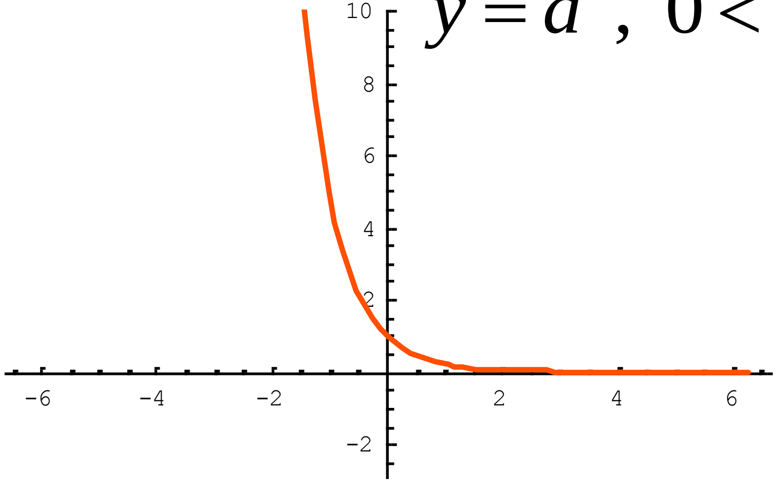
g) $\lim_{x\to\left(-3\pi/2\right)} \cot x = \lim_{x\to\left(-3\pi/2\right)} \dfrac{\cos x}{\sin x} = \dfrac{0}{1} = 0$

Limit and Exponential Functions

$y = a^x, \; a > 1$



$y = a^x, \; 0 < a < 1$



The above graph confirm that exponential functions are continuous everywhere.

$$\lim_{x \to c} a^x = a^c$$

Asymptotes

The line $y = L$ is called a horizontal asymptote of the curve $y = f(x)$ if eihter

$$\lim_{x\to\infty} f(x) = L \text{ or } \lim_{x\to-\infty} f(x) = L.$$

The line $x = c$ is called a vertical asymptote of the curve $y = f(x)$ if eihter

$$\lim_{x\to c^-} f(x) = \pm\infty \text{ or } \lim_{x\to c^+} f(x) = \pm\infty.$$

Examples

Find the asymptotes of the graphs of the functions

1. $f(x) = \dfrac{x^2+1}{x^2-1}$

(i) $\lim\limits_{x \to 1^-} f(x) = -\infty$

Therefore the line $x = 1$

is a vertical asymptote.

(ii) $\lim\limits_{x \to -1^-} f(x) = +\infty.$

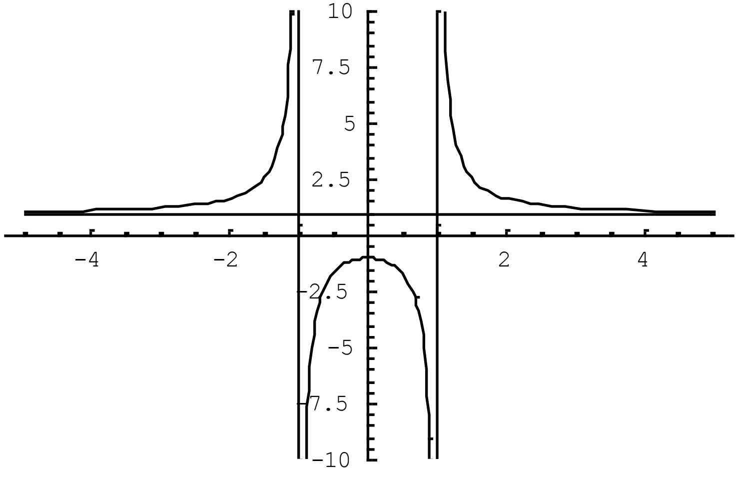
Therefore the line $x = -1$

is a vertical asymptote.

(iii) $\lim\limits_{x \to \infty} f(x) = 1.$

Therefore the line $y = 1$

is a horizonatl asymptote.



2. $f(x) = \dfrac{x-1}{x^2-1}$

(i) $\lim_{x \to 1} f(x) = \lim_{x \to 1} \left(\dfrac{x-1}{x^2-1} \right)$

$= \lim_{x \to 1} \left(\dfrac{x-1}{(x-1)(x+1)} \right) = \lim_{x \to 1} \left(\dfrac{1}{x+1} \right) = \dfrac{1}{2}.$

Therefore the line $x = 1$

is NOT a vertical asymptote.

(ii) $\lim_{x \to -1^+} f(x) = +\infty.$

Therefore the line $x = -1$

is a vertical asymptote.

(iii) $\lim_{x \to \infty} f(x) = 0.$

Therefore the line $y = 0$

is a horizonatl asymptote.