



BETA AND GAMMA FUNCTIONS

Beta function

The first eulerian integral $B(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1} dt$ where $m > 0$, $n > 0$ is called a Beta function and is denoted by $B(m, n)$.

The quantities m and n are positive but not necessarily integers.

Example:-

Properties of Beta Function

$$B(x, y) = B(y, x).$$

$$B(x, y) = \int_0^{\infty} \frac{t^{x-1}}{(1+t)^{x+y}} dt, \quad \operatorname{Re}(x) > 0, \operatorname{Re}(y) > 0$$

$$B(x, y) = B(x, y+1) + B(x+1, y)$$

$$xB(x, y+1) = yB(x+1, y)$$

$$B(x, y) = 2 \int_0^{\pi/2} (\sin \theta)^{2x-1} (\cos \theta)^{2y-1} d\theta, \quad \operatorname{Re}(x) > 0, \operatorname{Re}(y) > 0$$

$$B(x, y) = \frac{\Gamma(x) \Gamma(y)}{\Gamma(x+y)}$$

$$B(x, y) \cdot B(x+y, 1-y) = \frac{\pi}{x \sin(\pi y)},$$

Gamma function

The Eulerian integral $\Gamma(x) = \int_0^{\infty} t^{x-1} e^{-t} dt$, $x > 0$ is called gamma function and is denoted by $\Gamma(x)$

$$\begin{aligned}\text{Example:- } \Gamma(1) &= \int_0^{\infty} t^{1-1} e^{-t} dt \\ &= \lim_{n \rightarrow \infty} \int_0^n e^{-t} dt \\ &= \lim_{n \rightarrow \infty} -e^{-t} \Big|_0^n \\ &= \lim_{n \rightarrow \infty} \frac{-1}{e^n} - \frac{-1}{e^0} = \lim_{n \rightarrow \infty} 1 - \frac{1}{e^n} = 1\end{aligned}$$

Recurrence formulae for gamma function

$$\Gamma(x+1) = x\Gamma(x)$$

$$\Gamma(x+1) = \int_0^{\infty} t^x e^{-t} dt \quad \text{Use integration by parts.}$$

$$u = t^x \qquad dv = e^{-t} dt$$

$$du = xt^{x-1} dt \qquad v = -e^{-t}$$

$$\Gamma(x+1) = -t^x e^{-t} \Big|_0^{\infty} - \int_0^{\infty} (-e^{-t}) xt^{x-1} dt$$

$$\Gamma(x+1) = 0 + \int_0^{\infty} xt^{x-1} e^{-t} dt = x \int_0^{\infty} t^{x-1} e^{-t} dt$$

$$\therefore \Gamma(x+1) = x\Gamma(x)$$

Relation between gamma and factorial

$$\Gamma(n+1) = n!$$

Other results

$$\Gamma(1/2) = \int_{-\infty}^{\infty} e^{-u^2} du = \sqrt{\pi}$$

$$\Gamma(3/2) = \frac{1}{2}\Gamma(1/2) = \frac{\sqrt{\pi}}{2}$$

$$\Gamma(5/2) = \frac{3}{2}\Gamma(3/2) = \frac{3}{2} \frac{\sqrt{\pi}}{2} = \frac{3\sqrt{\pi}}{4}$$

$$\Gamma\left(n + \frac{1}{2}\right) = \frac{1 \cdot 3 \cdot 5 \cdots (2n - 1)}{2^n} \sqrt{\pi} \quad n = 1, 2, 3, \dots$$

Relation between beta and gamma function

$$B(a, b) = \int_0^1 x^{a-1}(1-x)^{b-1} dx.$$

Setting $x = y + \frac{1}{2}$ gives the more symmetric formula

$$B(a, b) = \int_{-1/2}^{1/2} \left(\frac{1}{2} + y\right)^{a-1} \left(\frac{1}{2} - y\right)^{b-1} dy.$$

Now let $y = \frac{t}{2s}$ to obtain

$$(2s)^{a+b-1} B(a, b) = \int_{-s}^s (s+t)^{a-1} (s-t)^{b-1} dt.$$

Multiply by e^{-2s} then integrate with respect to s , $0 \leq s \leq A$, to get

$$B(a, b) \int_0^A e^{-2s} (2s)^{a+b-1} ds = \int_0^A \int_{-s}^s e^{-2s} (s+t)^{a-1} (s-t)^{b-1} dt ds.$$

Take the limit as $A \rightarrow \infty$ to get

$$\frac{1}{2}B(a, b)\Gamma(a + b) = \lim_{A \rightarrow \infty} \int_0^A \int_{-s}^s e^{-2s}(s + t)^{a-1}(s - t)^{b-1} dt ds.$$

Let $\sigma = s + t$, $\tau = s - t$, so we integrate over

$$R = \{(\sigma, \tau) : \sigma + \tau \leq 2A, \sigma, \tau \geq 0\}.$$

Since $s = \frac{1}{2}(\sigma + \tau)$, $t = \frac{1}{2}(\sigma - \tau)$ the Jacobian determinant of the change of variables is

$$J = \begin{vmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{vmatrix} = \frac{1}{2}$$

so

$$\frac{1}{2}B(a, b)\Gamma(a + b) = \lim_{A \rightarrow \infty} \iint_R \frac{1}{2} e^{-(\sigma+\tau)} \sigma^{a-1} \tau^{b-1} d\tau d\sigma.$$

Thus

$$\begin{aligned} B(a, b)\Gamma(a + b) &= \int_0^\infty \int_0^\infty e^{-(\sigma+\tau)} \sigma^{a-1} \tau^{b-1} d\tau d\sigma \\ &= \int_0^\infty \int_0^\infty e^{-\sigma} \sigma^{a-1} e^{-\tau} \tau^{b-1} d\tau d\sigma \\ &= \left(\int_0^\infty e^{-\sigma} \sigma^{a-1} d\sigma \right) \left(\int_0^\infty e^{-\tau} \tau^{b-1} d\tau \right). \end{aligned}$$

Thus, we have... $B(x, y) = \frac{\Gamma(x) \Gamma(y)}{\Gamma(x + y)}$



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