Non-singular Linear Transformations
and
MATRIX REPRESENTATION OF A LINEAR TRANSFORMATION

SUBMITTED BY:
Ms. Harjeet Kaur
Associate Professor
Department of Mathematics
PGGCG – 11, Chandigarh
Definition: A linear transformation $T : V \to V$ is said to be **non-singular** if

\[ T(v) = 0 \implies v = 0 \]

i.e. $N(T) = \{0\}$

Definition: A linear transformation $T : V$ is said to be **singular** if $\exists$ some $v \in V$ s.t. $v \neq 0$ & $T(v) = 0$

i.e. $N(T)$ contains at least one **zero element**.
**Definition:** A linear transformation is an **isomorphism** if it is one-one and onto.

i.e. \( T : V \rightarrow W \) is an isomorphism if
(1) \( T \) is linear transformation.
(2) \( T \) is one-one.
(3) \( T \) is onto.

Then \( V \) and \( W \) are called **isomorphic**. We write \( V \cong W \)
THEOREM: \( V \cong W \iff \text{dim } V = \text{dim } W \)

Proof:- Let \( V \cong W \) and \( \text{dim } V = n \)
let \( B = \{ v_1, v_2, \ldots, v_n \} \) be a basis set for \( V \).

Claim: \( B_1 = \{ T(v_1), T(v_2), \ldots, T(v_n) \} \) is a basis for \( W \).

i.e. \( B_1 \) is linearly independent and the linear span of \( B_1 \) is \( W \).
Then \( \text{dim } W = n \).
conversely, let \( \text{dim} V = \text{dim} W = n \)

Let \( \{v_1, v_2, \ldots, v_n\} \) be a basis of \( V \)

and \( \{w_1, w_2, \ldots, w_n\} \) be a basis of \( W \).

To define \( T: V \rightarrow W \),

let \( v \in V \)

\( \exists \) unique scalars \( \alpha_1, \alpha_2, \ldots, \alpha_n \in F \)

Such that \( v = \alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_n v_n \)

Define \( T(v) = \alpha_1 w_1 + \alpha_2 w_2 + \ldots + \alpha_n w_n \)

i.e. \( T(\alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_n v_n) = \alpha_1 w_1 + \alpha_2 w_2 + \ldots + \alpha_n w_n \)

\( T \) is well-defined

(1) \( T \) is linear-transformation.

(2) \( T \) is one-one.

(3) \( T \) is onto.

Hence \( V \cong W \)
Theorem: Every n-dimensional vector space over the field F is isomorphic to the space $F^n$.

Proof:- Let $\dim V = n$

Let $B = \{v_1, v_2, \ldots, v_n\}$ be an ordered basis for $V$.

Define $T: V \to F^n$, as follows:

Let $v \in V$, $\exists$ unique scalars $\alpha_1, \alpha_2, \ldots, \alpha_n \in F$

$v = \alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_n v_n$.

Define $T(v) = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in F^n$.

$T$ is well-defined

To prove $V \cong F^n$

to prove that $T$ is an isomorphism.

(i) To prove $T$ is linear-transformation :-

$T(\alpha x + \beta y) = \alpha T(x) + \beta T(y)$ for all $x, y \in V$ and $\alpha, \beta \in F$

(ii) To prove $T$ is one-one:- $T(x) = T(y) \Rightarrow x = y$

(iii) To prove $T$ is onto:- Now for $(\alpha_1, \alpha_2, \ldots, \alpha_n) \in F^n$

$\exists \alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_n v_n = x \in V$

s.t $T(x) = (\alpha_1, \alpha_2, \ldots, \alpha_n)$

Hence $V \cong F^n$.  

Theorem: \( L(V, W) \) = the set all linear transformations from \( V(F) \) into \( W(F) \)
is a vector space over the field \( F \)
with vector addition and scalar multiplication defined by

\[
(T_1 + T_2)x = T_1(x) + T_2(x), \; \forall \; x \in V, \; and \; T_1, T_2 \in L(V, W)
\]

\[
(\alpha T_1)x = \alpha T_1(x) \; \forall \; x \in V \; and \; T_1 \in L(V, W) \; and \; \alpha \in F.
\]

if \( \text{dim} V = m \) \& \( \text{dim} W = n \) \; then \( \text{dim} \ L(V, W) = mn \)
Product of two linear transformations:-
Let U, V and W be three vector spaces over the same field and $T_1 : V \rightarrow W; T_2 : U \rightarrow V$ be two linear transformations. Then the composite mapping $T_1 T_2 : U \rightarrow W$ is defined by

$$(T_1 T_2)(x) = T_1[T_2(x)], \forall x \in U.$$ 

In general $T_1 T_2 \neq T_2 T_1$

e.g. Let $T_1, T_2 : \mathbb{R}^2 \rightarrow \mathbb{R}^2$

$T_1(a, b) = (a, 0), T_2(a, b) = (0, a)$

$T_1 T_2(a, b) = (0, 0), \quad T_2 T_1(a, b) = (0, a)$
U, V, W be three vector spaces over the same field F. Let T and T’ be linear transformations from U to V. Let S and S’ be linear transformations from V to W then

(i) \( S(T + T') = ST + ST' \)

(ii) \( (S + S')T = ST + S'T \)

(iii) \( \alpha(ST) = (\alpha S)T = S(\alpha T) \) for \( \alpha \in F \)
Let $V$ be a vector space over $F$ and $T: V \rightarrow V$ be a linear transformation, then $T$ is invertible if and only if $T$ is one–one and onto.

If $T$ is a linear operator on $V(F)$ and $T$ is invertible, then the inverse mapping $T^{-1}$ defined as

$$y_0 = T(x_0) \iff T^{-1}(y_0) = x_0$$

for each $x_0, y_0 \in V$ is a linear transformation.
Let $V(F)$ and $W(F)$ be two finite dimensional vector spaces such that $\dim V = \dim W$. If $T$ is a linear transformation from $V$ and $W$, then the followings are equivalent:

(i) $T$ is invertible
(ii) $T$ is non-singular i.e. the null space of $T = \{0\}$

Proof: $T: V \rightarrow W$ be invertible.

$\exists S: W \rightarrow V$ such that $ST = TS = I$

Let $T(v) = 0$

$\Rightarrow S(T(v)) = 0$

$\Rightarrow (ST)(v) = 0$

$\Rightarrow I(v) = 0$

$\Rightarrow v = 0$
Example: Let $T$ be the linear operator on $\mathbb{R}^3$ defined by

$$T(\ x, \ y, \ z) = (2x, \ 4x - y, \ 2x + 3y - z)$$

(i) show that $T$ is invertible (ii) Find $T^{-1}$.

Solution: To show $T$ is invertible i.e. $T$ is non-singular.

Let $(x, y, z) \in N(T)$

$$\Rightarrow T(x, y, z) = (0, 0, 0)$$

$$\Rightarrow (2x, 4x - y, 2x + 3y - z) = (0, 0, 0)$$

$$\Rightarrow 2x = 0, \ 4x - y = 0, \ 2x + 3y - z = 0$$

$$\Rightarrow x = 0, y = 0, z = 0$$

$$\Rightarrow N(T) = \{ (0, 0, 0) \}$$

hence $T$ is non-singular and so $T$ is invertible.
(ii) Let \( T^{-1}(r, s, t) = (x, y, z) \).
\[ \Rightarrow T(x, y, z) = (r, s, t) \]
\[ \Rightarrow (2x, 4x - y, 2x + 3y - z) = (r, s, t) \]
\[ \Rightarrow 2x = r, \ 4x - y = s, \ 2x + 3y - z = t \]
\[ \Rightarrow x = \frac{1}{2} r, \ y = 2r - s, \ z = 7r - 3s - t \]
\[ \Rightarrow T^{-1} (r, s, t) = (\frac{1}{2} r, 2r - s, 7r - 3s - t). \]
Let $V$ and $W$ be finite-dimensional vector spaces and $\dim V = n$ & $\dim W = m$

Let $B=\{x_1, x_2, \ldots, x_n\}$ be an ordered basis of $V$ and $B_1=\{y_1, y_2, \ldots, y_m\}$ be an ordered basis of $W$.

Let $T : V \rightarrow W$ be a linear transformation (for $x \in V$, $T(x)$ is uniquely expressible as linear combination of elements of $W$.)

$$T(x_1) = a_{11}y_1 + a_{21}y_2 + a_{31}y_3 + \ldots + a_{m1}y_m = \sum_{i=1}^{m} a_{i1}y_i$$
$$T(x_2) = a_{12}y_1 + a_{22}y_2 + a_{32}y_3 + \ldots + a_{m2}y_m = \sum_{i=1}^{m} a_{i2}y_i$$
$$T(x_3) = a_{13}y_1 + a_{23}y_2 + a_{33}y_3 + \ldots + a_{m3}y_m = \sum_{i=1}^{m} a_{i3}y_i$$
$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$

$$T(x_n) = a_{1n}y_1 + a_{2n}y_2 + a_{3n}y_3 + \ldots + a_{mn}y_m = \sum_{i=1}^{m} a_{in}y_i$$
\[ [T] = \begin{pmatrix}
  a_{11} & a_{12} & \cdots & a_{1n} \\
  \vdots & \ddots & & \vdots \\
  a_{m1} & a_{m2} & \cdots & a_{mn}
\end{pmatrix} \]

i.e. \([T:B, B_1] = (a_{ij})_{m \times n}\)
Thank You