# Non-singular Linear Transformations

and

# MATRIX REPRESENTATION OF A LINEAR TRANSFORMATION

**SUBMITTED BY:**

Ms. Harjeet Kaur

Associate Professor

Department of Mathematics

PGGCG – 11, Chandigarh

Definition: A linear transformation $T : V \rightarrow V$ is said to be **non-singular** if

$$\mathbf{T(v) = 0 \quad \Rightarrow \quad v = 0}$$

**i.e.** $\mathbf{N(T) = \{0\}}$

Definition: A linear transformation T : V is said to be **singular if $\exists$ some $v \in V$ s.t. $v \neq 0$ & $T(v) = 0$**

**i.e. N(T)** contains at least **one-zero element.**

**Definition:** A linear transformation is an **isomorphism** if it is one-one and onto.

i.e. $T : V \rightarrow W$ is an isomorphism if

**(1) T is linear transformation.**

**(2) T is one-one.**

**(3) T is onto.**

Then V and W are called **isomorphic.**

We write $\mathbf{V \cong W}$

**THEOREM:** **V ≅ W ⇔ dim V = dim W**

Proof:- Let V ≅ W and dim V=n

let B={ $v_{1,}\ v_{2,}$……………., $v_n$} be a basis set for V.

Claim: **$\boldsymbol{B_1}$= { T($\boldsymbol{v_1}$), T($\boldsymbol{v_2}$), ……………….,T($\boldsymbol{v_n}$)} is a basis for W**.

i.e. $B_1$ is linearly independent and the linear span of $B_1$ is W.

Then dim W = n.

conversly, let $\dim V = \dim W = n$

Let $\{v_1, v_2, \dots \dots \dots . . , v_n\}$ be a basis of V

and $\{w_1, w_2, \dots \dots \dots . . , w_n\}$ be a basis of W .

To define $T: V \to W$ ,

let $v \in V$

$\exists$ **unique** scalars $\alpha_1, \alpha_2, \dots \dots \dots , \alpha_n \in F$

Such that $v = \alpha_1 v_1 + \alpha_2 v_2 + \dots\dots\dots\dots + \alpha_n v_n$

Define $T(v) = \alpha_1 w_1 + \alpha_2 w_2 + \dots\dots\dots\dots + \alpha_n w_n$

i.e. $T(\alpha_1 v_1 + \alpha_2 v_2 + \dots\dots\dots\dots + \alpha_n v_n)$

$= \alpha_1 w_1 + \alpha_2 w_2 + \dots\dots\dots\dots + \alpha_n w_n$

T is **well-defined**

(1) T is linear-transformation.

(2) T is one-one.

(3) T is onto.

Hence **$V \cong W$**

Theorem: **Every n-dimensional vector space over the field F is isomorphic to the space $F^n$.**

Proof:- Let dimV = n

Let B =$\{v_{1\,,}\ v_{2\,,}$ ……………., $v_n\}$ be an **ordered basis** for V.

Define T: V → $F^n$, as follows:

Let v ∈ V, ∃ **unique** scalars $\alpha_1, \alpha_2, \ldots \ldots \ldots, \alpha_n \in$ F

v = $\alpha_1 v_1$ + $\alpha_2 v_2$+ …………+ $\alpha_n v_n$.

Define T(v) =( $\alpha_1$, $\alpha_2$, …………, $\alpha_n$) ∈ $F^n$.

T is **well-defined**

To prove **V ≅ $F^n$**

to prove that **T is an isomorphism**.

(i) To prove **T is linear-transformation** :-

T($\alpha x + \beta y) = \alpha T(x) + \beta T(y)$ for all x,y ∈ V and $\alpha, \beta$ ∈ F

(ii) To prove **T is one-on**e:- T(x) = T(y) ⇒ $x = y$

(iii) To prove **T is onto**:- Now for ($\alpha_{1,}\alpha_2$…………….,$\alpha_n$) ∈ $F^n$

∃ $\alpha_1 v_1$+$\alpha_2 v_2$+……………+$\alpha_n v_n$= x ∈ $V$

s.t T(x) = ( $\alpha_1$,$\alpha_{2\,,}$…………$\alpha_n$) *Hence* V ≅ $F^n$.

**Theorem: L(V,W)** = the set all linear transformations from V(F) into W(F)

is a **vector space** over the field F

with vector addition and scalar multiplication defined by

$(T_1+T_2)x = T_1(x) + T_2(x)$, $\forall$ x$\in V$, and $T_1, T_2 \in L(V,W)$

$(\alpha T_1)x = \alpha T_1(x)$ $\forall$ x $\in$ V and $T_1 \in$ L(V,W) and $\alpha$ $\in$ F.

if dimV =m & dimW = n then dim L(V,W)= mn

**Product of two linear transformations:-**

Let U,V and W Be three vector spaces over the same field and $T_1$ : V → W ; $T_2$ : U → V be two linear transformations. Then the composite mapping $T_1 T_2$: U → W is defined by

**($T_1 T_2$ ) (x) = $T_1[T_2$(x)]**, ∀ x∈ U.

In general $T_1 T_2 \neq T_2 T_1$

e.g. Let $T_1, T_2 : R^2 \rightarrow R^2$

$$T_1(a, b) = (a, 0) \quad , T_2(a, b) = (0, a)$$

$$T_1 T_2(a, b) = (0,0) , \qquad T_2 T_1(a, b) = (0, a)$$

U,V,W be three vector spaces over the same field F.

Let T and T' be linear transformations from U to V.

let S and S' be linear transformations from V to W

then

(i) $S(T + T') = ST + ST'$

(ii) $(S + S')T = ST + S'T$

(iii) $\alpha(ST) = (\alpha S)T = S(\alpha T)$ for $\alpha \in F$

Let V be a vector space over F
and T: V→V be a linear transformation ,
then T is invertible if and only if T is one –one and onto.

If T is a linear operator on V(F) and T is invertible ,
then the inverse mapping $T^{-1}$ defined as

$y_0$ = T($x_0$) ⇔ $T^{-1}(y_0)$ = $x_0$ for each $x_0$, $y_0$ ∈ V

is a linear transformation.

Let V(F) and W(F) be two finite dimensional vector spaces
such that dim V =dim W. If T is linear transformation from V and W, then the followings are equivalent:

(i) T is invertible

(ii) T is non – singular i.e. the null space of T = {O}

Proof: $T: V \to W$ be invertible.

$\exists\ S: W \to V$ such that $ST = TS = I$

Let T(v) = 0

$\Rightarrow S(T(v)) = 0$

. $\Rightarrow (ST)(v) = 0$

$\Rightarrow I(v) = 0$

$\Rightarrow v = 0$

Example: Let T be the linear operator on $R^3$ defined by

$$T(x, y, z) = (2x , 4x - y , 2x + 3y - z)$$

(i) show that T is invertible (ii) Find $T^{-1}$.

Solution: To show T is invertible i.e. T is non-singular.

Let $(x, y, z) \in N(T)$

$\Rightarrow T(x, y, z) = (o,o,o)$

$\Rightarrow (2x , 4x - y, 2x + 3y - z) = (0,0,0)$

$\Rightarrow 2x = 0, \quad 4x - y = 0, \quad 2x + 3y - z = 0$

$\Rightarrow x=0, y=0, z=0$

$\Rightarrow N(T) = \{ (0,0,0) \}$

hence T is non- singular and so T is invertible.

(ii) Let $T^{-1}$(r, s, t) = (x, y, z).

$\Rightarrow$ T(x, y, z) = (r, s, t)

$\Rightarrow$ ( 2x, 4x – y, 2x +3y –z) = (r, s, t)

$\Rightarrow$ 2x= r , 4x – y = s , 2x +3y –z = t

$\Rightarrow$ $x = \frac{1}{2}$ r , y = 2r – s , z = 7r -3s –t

$\Rightarrow$ $T^{-1}$ ( r,s,t) = ( $\frac{1}{2}$ r , 2r –s , 7r -3s – t).

Let V and W be finite-dimensional vector spaces and dimV = n & dimW = m

Let B={$x_1, x_2, \ldots \ldots .., x_n$} be an ordered basis of V and $B_1$={$y_1, y_2, \ldots \ldots .., y_m$} be an ordered basis of W.

Let T : V → W be a linear transformation

(for x ∈ V, T(x) is uniquely expressible as linear combination of elements of W.)

T($x_1$) = $a_{11}y_1 + a_{21}y_2 + a_{31}y_3 + \ldots\ldots + a_{m1}y_m = \sum_{i=1}^{m} a_{i1}y_i$

T($x_2$) = $a_{12}y_1 + a_{22}y_2 + a_{32}y_3 + \ldots\ldots + a_{m2}y_m = \sum_{i=1}^{m} a_{i2}y_i$

T($x_3$) = $a_{13}y_1 + a_{23}y_2 + a_{33}y_3 + \ldots\ldots + a_{m3}y_m = \sum_{i=1}^{m} a_{i3}y_i$

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T($x_n$) = $a_{1n}y_1 + a_{2n}y_2 + a_{3n}y_3 + \ldots\ldots + a_{mn}y_m = \sum_{i=1}^{m} a_{in}y_i$

$$[T] = \begin{pmatrix} a_{11} & a_{12} \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} \cdots & a_{mn} \end{pmatrix}$$

i.e. $[T{:}B, B_1] = (a_{ij})_{m\times n}$

Thank You