INTERPOLATION
OUTLINE

- Lagrange Interpolation
- Hermite Interpolation
- Divided Difference Interpolation
- Newton’s Forward/Backward Interpolation
- Gauss Forward/Backward Interpolation
- Stirling’s Formula
- Bessel’s Formula
WHAT IS INTERPOLATION?

Given \((x_0, y_0), (x_1, y_1), \ldots, (x_n, y_n)\), finding the value of ‘\(y\)’ at a value of ‘\(x\)’ in \((X_0, X_n)\) is called interpolation.
LAGRANGE POLYNOMIALS

- The formula used to interpolate between data pairs \((x_0, f(x_0)), (x_1, f(x_1)), \ldots, (x_n, f(x_n))\) is given by,

- Where the polynomial \(P_j(x)\) is given by,

\[
P(x) = \sum_{j=1}^{n} P_j(x)
\]

\[
P_j(x) = y_j \prod_{\substack{k=1 \\ k \neq j}}^{n} \frac{x - x_k}{x_j - x_k}
\]
LAGRANGE POLYNOMIALS

In general,

\[ P(x) = y_1 \frac{(x - x_2)(x - x_3)\ldots(x - x_n)}{(x_1 - x_2)(x_1 - x_3)\ldots(x_1 - x_n)} + \]
\[ + y_2 \frac{(x - x_1)(x - x_3)\ldots(x - x_n)}{(x_2 - x_1)(x_2 - x_3)\ldots(x_2 - x_n)} + \]
\[ + y_n \frac{(x - x_1)(x - x_2)\ldots(x - x_{n-1})}{(x_n - x_1)(x_n - x_2)\ldots(x_n - x_{n-1})} + \ldots \]
Hermite Interpolation

In Hermite Interpolation, the interpolating polynomial $p(x)$ coincides with $f(x)$ as well as $p'(x)$ coincides with $f'(x)$ at the interpolating points.

The formula is:

$$p(x) = \sum_{k=0}^{n} \left\{ 1 - 2L'_k(x_k)(x - x_k) \right\} [L_k(x)]^2 f(x_k) + \sum_{k=0}^{n} (x - x_k)[L_k(x)]^2 f'(x_k)$$

Where $L_k(x)$ is Lagrange Polynomial of degree $n$
NEWTONS DIVIDED DIFFERENCE

What is divided difference?

\[ f[x_0, x_1] = \frac{f[x_1] - f[x_0]}{x_1 - x_0} \]

\[ f[x_0, x_1, x_2] = \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_1} \]

\[ f[x_0, x_1, \ldots, x_{k-1}, x_k] = \frac{f[x_1, x_2 - x_k] - f[x_0, \ldots, x_{k-1}]}{x_k - x_0} \]

for \( k = 3, 4, \ldots, n \).

These \( I^{st} \), \( II^{nd} \)… and \( k^{th} \) order differences are denoted by \( \Delta f \), \( \Delta^2 f \), …, \( \Delta^k f \).

The divided difference interpolation polynomial is:

\[ P(x) = f(x_0) + (x - x_0) f[x_0, x_1] + (x - x_0)(x - x_1) f[x_0, x_1, x_2] + \ldots + (x - x_0)(x - x_1)\ldots(x - x_{n-1}) f[x_0, x_1, \ldots, x_n] \]
INTERPOLATION FOR EQUALLY SPACED POINTS

Let \((X_0, Y_0), (X_1, Y_1), \ldots, (X_n, Y_n)\) be the given points with 
\(X_{i+1} = X_i + h, \ i = 0, 1, 2, \ldots, (n-1)\).

- **Finite Difference Operators**
  
  - Forward difference operator
    \[ \Delta f(x_i) = f(x_i + h) - f(x_i) \]
  
  - Backward difference operator
    \[ \nabla f(x_i) = f(x_i) - f(x_i - h) \]
  
  - Central difference operator
    \[ \delta f(x_i) = f\left(x_i + \frac{h}{2}\right) - f\left(x_i - \frac{h}{2}\right) \]
  
  - Shift operators
    \[ E f(x_i) = f(x_i + h) \]
    \[ E^r f(x_i) = f(x_i + rh) \]
NEWTON FORWARD INTERPOLATION

For convenience we put $p = \frac{x - x_0}{h}$ and $f_0 = y_0$. Then we have

$$f(x_0 + ph) = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!} \Delta^2 y_0 + \ldots + \frac{p(p-1)\ldots(p-n-1)}{n!} \Delta^n y_0$$

NEWTON BACKWARD INTERPOLATION

For convenience we put $p = \frac{x - x_n}{h}$ and $f_0 = y_0$. Then we have

$$f(x_n + ph) = y_n + p\nabla y_n + \frac{p(p+1)}{2!} \nabla^2 y_n + \ldots + \frac{p(p+1)\ldots(p+n-1)}{n!} \nabla^n y_n$$
INTERPOLATION USING CENTRAL DIFFERENCES

Suppose the values of the function \( f(x) \) are known at the points \( a - 3h, a - 2h, a - h, a, a + h, a + 2h, a + 3h, \ldots \) etc. Let these values be \( y_{-3}, y_{-2}, y_{-1}, y_0, y_1, y_2, y_3, \ldots \), and so on. Then we can form the central difference table as:

<table>
<thead>
<tr>
<th>x</th>
<th>f(x)</th>
<th>( \Delta f )</th>
<th>( \Delta^2 f )</th>
<th>( \Delta^3 f )</th>
<th>( \Delta^4 f )</th>
<th>( \Delta^5 f )</th>
<th>( \Delta^6 f )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( a-3h )</td>
<td>( y_{-3} )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>( \Delta y_{-3} )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( a-2h )</td>
<td>( y_{-2} )</td>
<td></td>
<td>( \Delta^2 y_{-3} )</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>( \Delta y_{-2} )</td>
<td>( \Delta^2 y_{-3} )</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( a-h )</td>
<td>( y_{-1} )</td>
<td></td>
<td></td>
<td>( \Delta^3 y_{-3} )</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>( \Delta y_{-1} )</td>
<td>( \Delta^2 y_{-2} )</td>
<td>( \Delta^3 y_{-3} )</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( a )</td>
<td>( y_0 )</td>
<td>( \Delta^2 y_{-1} )</td>
<td>( \Delta^3 y_{-2} )</td>
<td>( \Delta^4 y_{-3} )</td>
<td>( \Delta^5 y_{-3} )</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>( \Delta y_0 )</td>
<td>( \Delta^3 y_{-1} )</td>
<td>( \Delta^4 y_{-2} )</td>
<td>( \Delta^5 y_{-2} )</td>
<td>( \Delta^6 y_{-3} )</td>
<td></td>
</tr>
<tr>
<td>( a+h )</td>
<td>( y_1 )</td>
<td>( \Delta^2 y_0 )</td>
<td>( \Delta^3 y_{-1} )</td>
<td>( \Delta^4 y_{-1} )</td>
<td>( \Delta^5 y_{-2} )</td>
<td>( \Delta^6 y_{-3} )</td>
<td>( \Delta^7 y_{-3} )</td>
</tr>
<tr>
<td></td>
<td></td>
<td>( \Delta y_1 )</td>
<td>( \Delta^3 y_0 )</td>
<td>( \Delta^4 y_{-1} )</td>
<td>( \Delta^5 y_{-2} )</td>
<td>( \Delta^6 y_{-3} )</td>
<td>( \Delta^7 y_{-3} )</td>
</tr>
<tr>
<td>( a+2h )</td>
<td>( y_2 )</td>
<td>( \Delta^2 y_1 )</td>
<td>( \Delta^3 y_0 )</td>
<td>( \Delta^4 y_{-1} )</td>
<td>( \Delta^5 y_{-2} )</td>
<td>( \Delta^6 y_{-3} )</td>
<td>( \Delta^7 y_{-3} )</td>
</tr>
<tr>
<td>( a+3h )</td>
<td>( y_3 )</td>
<td>( \Delta^3 y_0 )</td>
<td>( \Delta^4 y_{-1} )</td>
<td>( \Delta^5 y_{-2} )</td>
<td>( \Delta^6 y_{-3} )</td>
<td>( \Delta^7 y_{-3} )</td>
<td>( \Delta^8 y_{-3} )</td>
</tr>
</tbody>
</table>

We can relate the central difference operator \( \delta \) with \( \Delta \) and \( E \) using the operator relation \( \delta = \Delta E^{1/2} \).
GAUSS FORWARD INTERPOLATION FORMULA

• The value $p$ is measured forwardly from the origin and $0 < p < 1$.

• The above formula involves odd differences below the central horizontal line and even differences on the line. This is explained in the following figure.

• Formula is:

$$y_x = y_0 + \left( \frac{u}{1} \right) \Delta y_0 + \left( \frac{u}{2} \right) \Delta^2 y_{-1} + \left( \frac{u+1}{3} \right) \Delta^3 y_{-1} + \left( \frac{u+1}{4} \right) \Delta^4 y_{-2} + \left( \frac{u+2}{5} \right) \Delta^5 y_{-2} + \ldots$$

where $$\binom{u}{r} = \frac{u(u-1)(u-2)\ldots (u-r+1)}{r!}$$
GAUSS BACKWARD INTERPOLATION FORMULA

- The value $p$ is measured forwardly from the origin and $-1 < p < 0$.

- The above formula involves odd differences above the central horizontal line and even differences on the line.

- Formula is:

$$y_x = y_0 + \left( \begin{array}{c} u \\ 1 \end{array} \right) \Delta y_{-1} + \left( \begin{array}{c} u + 1 \\ 2 \end{array} \right) \Delta^2 y_{-1} + \left( \begin{array}{c} u + 1 \\ 3 \end{array} \right) \Delta^3 y_{-2} + \left( \begin{array}{c} u + 2 \\ 4 \end{array} \right) \Delta^4 y_{-2} + \left( \begin{array}{c} u + 3 \\ 5 \end{array} \right) \Delta^5 y_{-3} + \ldots \ldots$$

where $\left( \begin{array}{c} u \\ r \end{array} \right) = \frac{u(u-1)(u-2)\ldots(u-r+1)}{r!}$
STIRLING’S FORMULA

This formula gives the average of the values obtained by Gauss forward and backward interpolation formulae. For using this formula we should have \(-\frac{1}{2} < p < \frac{1}{2}\).

We can get very good estimates if \(-\frac{1}{4} < p < \frac{1}{4}\).
The formula is:

\[
y = y_0 + u \left( \frac{\Delta y_0 + \Delta y_{-1}}{2} \right) + \frac{u^2}{2!} \Delta^2 y_{-1} + \frac{u(u^2 - 1)}{3!} \left[ \frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2} \right] + \frac{u^2(u^2 - 1)}{4!} \Delta^4 y_{-2} + \frac{u(u^2 - 1)(u^2 - 4)}{5!} \left[ \frac{\Delta^5 y_{-2} + \Delta^5 y_{-3}}{2} \right] + \ldots
\]

where \( u = \frac{x - x_0}{h} \)
BESSEL’S INTERPOLATION FORMULA

- This formula involves means of even difference on and below the central line and odd difference below the line.

- The formula is:

\[
y = \frac{1}{2} (y_0 + y_1) + \left( u - \frac{1}{2} \right) \Delta y_0 + \frac{u(u-1)}{2!} \cdot \frac{1}{2} [\Delta^2 y_{-1} + \Delta^2 y_0] + \frac{(u - \frac{1}{2})u(u-1)}{3!} \Delta^3 y_{-1}
\]

\[
+ \frac{u+1)u(u-1)(u-2)}{4!} \cdot \frac{1}{2} [\Delta^4 y_{-1} + \Delta^4 y_{-2}] + ........
\]

where \( u = \frac{x-x_0}{h} \)

- Bessel’s formula gives better result for \( \frac{1}{4} < u < \frac{3}{4} \)
THANK YOU

Submitted by:
Rakesh Kumar,
(Deptt. Of Mathematics),
P.G.G.C.G, Sec-11,
Chandigarh.